Elastic analysis of Shrink-fitted Thick FGM Cylinders Based on Linear Plane Elasticity Theory

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ABSTRACT

Nowadays, functionally graded materials (FGMs) are widely used in many industrial, aerospace and military fields. On the other hand, the interest in the use of shrink-fitted assemblies is increasing for designing composite tubes, high-pressure vessels, rectors and tanks. Although extensive researches exist on thick-walled cylindrical shells, not many researches have been done on shrink-fitted thick FGM cylinders. In this paper, an analytical formulation for shrink-fitted of axisymmetric thick-walled FGM cylinders based on the linear plane elasticity theory is presented. The stresses and displacement fields in thick cylindrical shells are calculated using the real, repeated and complex roots of characteristic equation. The displacements and stresses resulted are depicted for a case study. The results show that the material composition variation had evident effects on shrink-fitting pressure in the intersection area of two fitted tubes. The value of this pressure affects radial and hoop stress distribution in FGM circular cylinders walls.

1. Introduction

Functionally Graded Materials (FGMs) are advanced composite materials in which two or more different material ingredients change continuously and gradually. This results in elimination of interface problems, and consequently, uniformity of stress and temperature distribution. FGM materials are made from ceramic and metal. Usually, the composition varies from a ceramic surface to a metal surface. These changes are considered as power or exponential functions in the thickness directions [1-3].

To date many theorists and practitioners have intensively studied shell structures (eg, cylindrical, spherical, etc.). Axisymmetric thick-walled cylindrical shells are common structural elements in many engineering applications, including pressure vessels, submarine hulls, rocket chambers, fuel storage tanks and many other structures. In order to optimize the weight, mechanical strength, displacement and stress distribution of a shell structure one approach is to use shells with functionally graded materials [4-8].

Shrink fitting is a process in which an interference fit is achieved by a relative size change after assembly. This is usually achieved by heating or cooling one component before assembly and allowing it to return to the ambient temperature after assembly [9, 10]. Heating the outer part may take a long time, and in some cases metallurgical reasons may prevent heating altogether. However, cooling the inner part will not affect the property of the material, and can be achieved much faster. Shrink fitting is a method used in making mechanical joints when axles, bushings and similar are to be fitted inside a part with a hole. Shrink-fitting is an effective way of assembling machine elements such as a gear to a shaft to transmit torque. In addition, shrink-fitted assemblies are widely used for steel rolling in the production of I-beams and rail sections. The advantage of a shrink-fitted assembly is that it is simple to manufacture, has high integrity...
and is capable of being subject to high loads. Shrink fitting of layered cylinders is a way of extending the fatigue lifetimes [11-15].

In 1852 for the first time, Lame’ proposed the exact solution of axisymmetric thick-walled cylindrical shells made of homogeneous and isotropic material [16]. He used plain elasticity theory to obtain the stress distribution. Lehnitsky in 1950 formulated the elasticity theory of composite structures [17]. In 1984, Niino et al. presented the concepts of FGM materials [18]. In 2006 Hongjun et al. derived the exact solution of FGM hollow cylinders in the state of plane strain with exponential function of elasticity modulus along the thickness of the cylinder [8]. In 2012, Ghannad and Zamani Nejad proposed a complete elastic solution of pressurized thick cylindrical shells made of heterogeneous FG materials [6]. They used plane elasticity theory. In 2018, Pourasghar et al. analyzed the three-dimensional thermoelastic deformation of FGM cylindrical shells subjected to thermal load [19].

During the past years, many scholars have studied the stress and strain distributions in hollow cylindrical shells made of FG materials but to the best of our knowledge, there is no complete analytical formulation for shrink-fitted thick FGM cylinders with a power variation of material properties. In 2006, Jehed et al. [13] studied the fatigue life time of multi-layer shrink-fitted cylinders made of homogeneous materials. In 2012, Sharifi et al. investigated optimum design of shrink-fit multi-layer compound cylinders made of homogeneous materials employing an analytical method [15]. In 2015, Arslan and Mack [20] presented an analytical formulation for rotating shrink-fitted cylinders with considering thermal effects. In their study the outer cylinder is made of FG material and the inner part is a homogenous shaft. They only considered the real distinct roots of the characteristic function. In 2017, Apatay et al. [21] investigated elastic-plastic design of a rotating shrink fit with functionally graded hub using semi-analytical formulation. Motivated by the aforementioned remarks the focus of this research is to develop a complete analytical formulation for the behavior of shrink-fitted thick FGM cylinders based on linear plain elasticity theory. The final formulation is a closed-form solution of the differential equations therefore a purely analytical discussion of the problem is possible. The presented formulation is general as it considers all the possible roots of the characteristic function. So it can be used for any grading index and any end boundary condition. The remaining of this article is organized as follows: In section “2,” analytical model is derived. In section “3,” a case study is presented. Finally, in section “2,” concluding remarks are made.

2. Formulation

According to the plane elasticity theory, it is assumed that all the plane sections perpendicular to the central axis of cylinder, after applying pressure, remain flat and perpendicular to the central axis. The resultant deformations are symmetric with respect to the central axis and their values do not change along the length of the cylinder. Thus, the radial displacement $u_r$ is only a function of radius $r$ and both longitudinal stress $\sigma_r$ and strain $\varepsilon_r$ have constant values and normal stresses are principal stresses [22].

$$\sigma_0 > \sigma_x > \sigma_r \quad (1)$$

The stress equation in the absence of body forces is defined by Eq. (2).

$$\frac{d\sigma_r}{dr} - \frac{\sigma_r - \sigma_0}{r} = 0 \quad (2)$$

Moreover, strain-displacement relation for axisymmetric case is defined as

$$\varepsilon_r = \frac{du_r}{dr}, \quad \varepsilon_0 = \frac{u_r}{r} \quad (3)$$

And for isotropic and heterogeneous material the stress is related to the strain as

$$\begin{bmatrix} \sigma_r \\ \sigma_0 \end{bmatrix} = E(r) \begin{bmatrix} A & B \\ B & A \end{bmatrix} \begin{bmatrix} \varepsilon_r \\ \varepsilon_0 \end{bmatrix} \quad (4)$$

Where $A$ and $B$ are functions of Poisson’s ratio and they are defined based on boundary conditions. In this paper, we assume that the Young’s modulus is a power function of radius [23].

$$E(r) = E_i \left(\frac{r}{n_1}\right)^n = E_i \tilde{r}^n \quad (5)$$

Where $E_i$ and $n_1$ are Young’s modulus and radius at inner side of the cylinder. $n$ is heterogeneity constant of material. Substituting Eqs. (3) and (4) in Eq. (2), one can obtain

$$r^2 \frac{d^2 u_r}{dr^2} + (n+1)r \frac{du_r}{dr} + \left(\frac{B}{A} \right) u_r = 0 \quad (6)$$
Eq. (6) is a homogeneous, linear second order ODE (Cauchy-Euler equation), in which the roots of the characteristic polynomial

\[ m^2 + nm + \left( \frac{nB}{A} - 1 \right) = 0 \]  

(7)
could be real distinct, repeated and complex. The roots of the characteristic polynomial are

\[ m_{1,2} = -\frac{n \pm \sqrt{\Delta}}{2}, \quad \Delta = n^2 - 4 \left( \frac{nB}{A} - 1 \right) \]  

(8)

2.1. Real distinct roots

In this case, \( \Delta > 0 \), the general solution of Eq. (6) is

\[ u_r(r) = C_1 r^{m_1} + C_2 r^{m_2} \]  

(9)

Using Eqs. (3) and (4), we get

\[ \sigma(r) = r^n \left[ C_1 (Am_1 + B)r^{m_1 - 1} + C_2 (Am_2 + B)r^{m_2 - 1} \right] \]  

(10)

![Fig. 1. Schematic model of inner and outer cylinders and their notations.](image)

According to figure 1, we can consider the following loading conditions. It is assumed that both members have the same length.

**Outer cylinder:**

\[ \sigma_r \bigg|_{r=R_1} = -P, \quad \sigma_r \bigg|_{r=R_2} = 0 \]  

(11)

\[ \sigma_r \bigg|_{r=R_3} = 0, \quad \sigma_r \bigg|_{r=R_0} = -P \]

(12)

Therefore based on Eqs. (10), (11) and (12), the constants \( C_1 \) and \( C_2 \) can be obtained. Eqs. (13) and (14) show the constants \( C_1 \) and \( C_2 \) for outer and inner cylinders respectively.

\[ C_1 = -\frac{Pr_0}{E_i \left( Am_1 + B \right) \left( k^{m_2} - k^{m_1} \right)} \]  

(13)

\[ C_2 = -\frac{Pr_0}{E_i \left( Am_2 + B \right) \left( k^{m_2} - k^{m_1} \right)} \]  

(14)

where

\[ k = \frac{n}{r_i} \]

(15)

Substituting \( C_1 \) and \( C_2 \) in Eqs. (4) and (9), we have the following equations.

**Outer cylinder:**

\[ \sigma_r = \frac{Pr^{n-1}}{k^{m_1} - k^{m_2}} \left( k^{m_2} r^{m_1 - 1} - k^{m_1} r^{m_2} \right) \]  

(16)

\[ \sigma_0 = \frac{Pr^{n-1}}{k^{m_1} - k^{m_2}} \left( A + B m_1 k^{m_2} - m_1 \right) \left( A + B m_2 k^{m_1} - m_1 \right) \left( A + B m_2 k^{m_1} - m_2 \right) \]  

(17)

\[ u_r(r) \bigg|_0 = \frac{Pr_0}{E_i \left( k^{m_1} - k^{m_2} \right)} \left( k^{m_2} r^{m_1} - k^{m_1} r^{m_2} \right) \]  

(18)

**Inner cylinder:**

\[ \sigma_r = \frac{P}{k^{m_2} - k^{m_1}} \left( r^{n+m_1} - r^{n+m_2} \right) \]  

(19)
Now we should deal with the contact pressure $P$. Prior to assembly, the outer radius of the inner cylinder was larger than the inner radius of the outer cylinder by the radial interference $\delta$.

$$\delta = u(r = r_1)|_0 + u(r = r_2)|_0$$

Substituting Eqs. (18) and (21) into (22), the contact pressure can be obtained.

2.2. Repeated roots

In this case, $\Delta = 0$, the general solution of Eq. (6) is

$$u_i(r) = (C_i + C_1 r) r^m$$

Following the same steps as section 2.1, we achieve to the Eqs. (24) to (31).

Outer cylinder:

$$C_1 = \frac{P \left[ A + (Am + B) ln r_0 \right]}{E_i k^2 (Am + B)^2 (ln k) r^{m-1}}$$

$$C_2 = \frac{P}{E_i (Am + B)(ln k) r^{m-1}}$$

$$\sigma_r = -\frac{p r^{(m+n-1)}}{ln k} \left[ \frac{ln k}{r} \right]$$

$$\sigma_0 = \frac{p r^{(m+n-1)}}{ln k} \left[ \frac{B^2 - A^2}{(Am + B)^2} \right] \left[ \frac{Am + B}{(Am + B)^2} \right]$$

$$u_i(r)|_0 = \frac{p r^m}{E_i (Am + B)(ln k) r^{m-1}}$$

Inner cylinder:

$$\sigma_r = \frac{p r^{(m+n-1)}}{sin (y ln k)} \left( y ln k \right)$$

2.3. Complex roots

In this case, $\Delta < 0$, so we have

$$m_{1,2} = \frac{z \pm iy}{2} = \frac{a}{2}, \quad y = \frac{\sqrt{-\Delta}}{2}$$

Therefore the general solution of Eq. (6) is

$$u_i(r) = [C_1 cos(y ln r) + C_2 sin(y ln r)] r^m$$

Again following the same steps as section 2.1, we achieve to the Eqs. (34) to (42).

Outer cylinder:

$$C_1 = \frac{p r^{(1-z)}}{E_i D} \left[ (Az + B) sin (y ln r_0) + Ay cos (y ln r_0) \right]$$

$$C_2 = \frac{p r^{(1-z)}}{E_i D} \left[ (Az + B) cos (y ln r_0) + Ay sin (y ln r_0) \right]$$

where

$$D = \left[ (Az + B)^2 + A^2 y^2 \right] sin(y ln k)$$

$$\sigma_r = \frac{p r^{(z+n-1)}}{sin (y ln k)} \left( y ln k \right)$$
\[\begin{align*}
\sigma_0 &= -\frac{p_{r0}^{z+n-1}}{D} \left\{ (A^2 - B^2) y \cos \left( \frac{y ln k r}{r} \right) \right. \\
&\quad \left. + \left[ AB (z^2 + y^2 + 1) + (A^2 + B^2) z \right] \sin \left( \frac{y ln k r}{r} \right) \right\} \\
&= \frac{Pr_{0}^{z+n-1}}{E(k^{z+n-1})} \left( Az + B \right) \sin \left( \frac{y ln k r}{r} \right) + Ay \cos \left( \frac{y ln k r}{r} \right)
\end{align*}\] (37)

Inner cylinder:

\[\begin{align*}
C_1 &= -\frac{Pr_{0}^{1-z}}{k^{z+n-1} D} \left( (Az + B) \sin \left( \frac{y ln r_i}{r_i} \right) + Ay \cos \left( \frac{y ln r_i}{r_i} \right) \right)
\end{align*}\] (39)

\[\begin{align*}
\sigma_r &= -\frac{Pr_{0}^{z+n-1}}{\sin \left( \frac{y ln h k}{r} \right) k^{z+n-1}} \sin \left( \frac{y ln h}{r} \right)
\end{align*}\] (40)

\[\begin{align*}
\sigma_0 &= -\frac{p_{r0}^{z+n-1}}{k^{z+n-1} D} \left\{ (B^2 - A^2) \cos \left( \frac{y ln h}{r} \right) + \left[ AB (z^2 + y^2 + 1) + (A^2 + B^2) z \right] \sin \left( \frac{y ln h}{r} \right) \right\}
\end{align*}\] (41)

\[\begin{align*}
u_0(r) \bigg|_1 &= \frac{Pr_{0}^{z+n-1}}{E(k^{z+n-1})} \left( (Az + B) \sin \left( \frac{y ln r_i}{r_i} \right) + Ay \cos \left( \frac{y ln r_i}{r_i} \right) \right)
\end{align*}\] (42)

3. Results and discussions

In this section as a case study, we consider two thick cylinders, which their characteristics are tabulated in table 1. Here we consider that for both of the cylinders the end boundary conditions are free-free. Eq. (43) shows the constants A and B for these boundary conditions. The analytical solution is carried out by writing an appropriated program in MATLAB.

\[\sigma_x = 0 \; , \; \varepsilon_x \neq 0 \quad A = \frac{1}{1 - v^2} \; , \; B = \frac{1}{1 - v^2} \] (43)

<table>
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<tr>
<th>Table 1. Geometrical and mechanical characteristics</th>
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Fig. 2 shows the distribution of the elasticity modulus along the cylinders thickness. In this case study the inner surface of both cylinders is softer. Figs. 3 and 4 show the distribution of radial and hoop stresses along the outer cylinder thickness respectively. It can be seen from Fig. 3 that by increasing the heterogeneity constant \( n \) from 2 to 2 the radial stress increases, which means the contact pressure value, is increasing too. Fig. 4 shows that the hoop stress for negative values of heterogeneity constant, the behavior of FG material is similar to homogeneous material.

Figs. 5 and 6 show the distribution of radial and hoop stresses along the inner cylinder thickness respectively. It is obvious that the distribution of the hoop stress in both cylinders strongly depends on the value and sign of the heterogeneity constant \( n \).

Next, we investigate the radial displacement in both cylinders. Figs. 7 and 8 show the radial displacement for the outer and inner cylinders respectively. It can be concluded from Fig. 7 that as the heterogeneity constant \( n \) increases the radial displacement increases too. According to Fig. 8 the behavior of inner cylinder is complicated. For \( n \leq 1 \) the behavior of FG material is similar to the homogeneous material but for \( n > 1 \) the behavior is completely different.
Fig 2. Distribution of elasticity modulus in the outer cylinder

Fig 3. Distribution of radial stress in the outer cylinder

Fig 4. Distribution of hoop stress in the outer cylinder

Fig 5. Distribution of radial stress in the inner cylinder

Fig 6. Distribution of hoop stress in the inner cylinder

Fig 7. Distribution of radial displacement in the outer cylinder
It is worth noting that in the case of homogeneous material \((n=0)\) we can achieve to the conventional relations \([22, 24]\) for hollow thick-walled cylinders under uniform internal and external pressure. Our present results for homogenous and isotropic material are in coincidence with reference \([25]\).

4. Conclusions and future work

In this paper, based on plane elasticity theory, an analytical solution has been obtained for shrink-fitted axisymmetric hollow thick-walled cylindrical shells made of FG materials and the effects of compositional gradient exponent \(n\) of FG circular cylinder were investigated. The variation of FGM properties is supposed to be a power function along the thickness of cylinders. The results show that with increasing the softness (elastic module) of intersection area of fitted cylinders, the shrink-fit pressure reduced. Radial stress in inner surface of outer cylinder and outer surface of inner cylinder are equal to the shrink-fit pressure and in free internal and external surfaces of each cylinder are equal to zero. Hoop stress distribution is also dependent to the shrink-fit pressure. This stress has a variable direction and magnitude in cylinders. For \(n=2, 1.0\) and \(1\) the hoop stress in inner and outer cylinders have values close together while for \(n=2\) the hoop stress in inner cylinder are more than outer one. Future work will involve the effects of the temperature on the elastic behavior of shrink-fitted thick FGM cylinders.

5. References


