Nonlinear Magneto-Nonlocal Vibration Analysis of Coupled Piezoelectric Micro-Plates Reinforced with Agglomerated CNTs

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1. Introduction

Nanocomposites hold the promise of advances that exceed those achieved in recent decades in composite materials. The nanostructure created by a nanophase in polymer matrix represents a radical alternative to the structure of conventional polymer composites. These complex-hybrid materials integrate the predominant surfaces of nanoparticles and the polymeric structure into a novel nanostructure, which produces critical fabrication and interface implementations leading to extraordinary properties [1]. PVDF is an ideal piezoelectric matrix due to characteristics including flexibility in thermoplastic conversion techniques, excellent dimensional stability, abrasion and corrosion resistance, high strength, and capability of maintaining its mechanical properties at elevated temperature. It has therefore found multiple applications in nanocomposites in a wide range of industries including oil and gas, petrochemical, wire and cable, electronics, automotive, and construction. Boron nitride nanotubes (BNNTs) used as the matrix reinforcing, apart from having high mechanical, electrical and chemical properties, present more resistant to oxidation than other conventional nano reinforcing such as carbon nanotubes (CNTs). Hence, they are used for high-temperature applications [2-6]. Both PVDF and BNNT are smart materials since they have piezoelectric property.

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Piezoelectricity is a classical discipline traced to the original work of Jacques and Pierre Curie around 1880. This phenomenon describes the relations between mechanical strains on a solid and its resulting electrical behavior resulting from changes in the electric polarization. One can create an electrical output from a solid resulting from mechanical strains or can create a mechanical distortion resulting from the application of an electrical perturbation [7]. Piezoelectric materials have been used to manufacture various sensors, conductors, actuators, etc. in fact, they have become one of the smart materials nowadays [8].

Regarding research development into the use of smart nanocomposite (which PVDF and BNNT are as matrix and reinforcer, respectively), Mosallaie Barzoki et al. [8] investigated electro-thermo-mechanical torsional buckling of a piezoelectric polymeric cylindrical shell reinforced by DWBNNTs with an elastic core. They concluded that the higher the in-fill core, the higher is dimensionless critical torsional buckling load. In another research, Mosallaie Barzoki et al. [9] studied nonlinear buckling response of embedded piezoelectric cylindrical shell reinforced with BNNT under electro-thermo-mechanical loadings using harmonic differential quadrature method (HDQM). They found that the critical buckling load increases when the piezoelectric effect is considered. Ghorbanpour et al. [10] carried out nonlinear vibration and stability of a smart composite micro tube made of PVDF reinforced by BNNTs embedded in an elastic medium under electro-thermo-loadings is investigated. They concluded that the stability of the system is strongly dependent on the imposed electric potential and the volume percent of BNNTs reinforcement.

In recent years, small scale effect in micro and nano applications of the beam, plate, and shell type structures has been utilized on the basis of nonlocal elasticity theory which was initiated in the papers of Eringen [11-13]. He regarded the stress state at a given point as a function of the strain states of all points in the body, while the local continuum mechanics assumes that the stress state at a given point depends uniquely on the strain state at the same point. Shen et al. [14] investigated the nonlocal plate model for nonlinear vibration of single-layer graphene sheets (SLGS) in thermal environments. Their results showed that with properly selected small scale parameters and material properties, the nonlocal plate model can provide a remarkably accurate prediction of the graphene sheet behavior under nonlinear vibration in the thermal environment. Pradhan and Kumar [15] reported vibration analysis of orthotropic graphene sheets using nonlocal elasticity theory and DQM. Their results indicated that the nonlocal effect increases as size of graphene sheet is decreased. Amir [16] studied Orthotropic patterns of visco-Pasternak foundation in nonlocal vibration of orthotropic graphene sheet under thermo-magnetic fields based on new first-order shear deformation theory. The results indicate that the stability of single-layer graphene sheet is strongly dependent on applied magnetic field. Ghorbanpour et al. [17] studied Pasternak foundation effect on the axial and torsional waves propagation in the embedded double-walled carbon nanotubes (DWCNTs) using nonlocal elasticity cylindrical shell theory. They concluded that the frequencies are dependent to small scale coefficient and shear modulus of the elastic medium.

The above studies on the nanostructures are on the basis of the nonlocal elasticity theory, which is not proper for direct use in the piezoelectric materials. Recently, Eringen’s nonlocal elasticity theory was extended by Zhou et al. [18-20] for the piezoelectric materials. In the nonlocal piezoelectric materials, the stress state and the electric displacement at a given point are, respectively, as a function of the strain state and electric potential of all points in the body. Ke et al. [21] employed nonlocal piezoelectric model to nonlinear vibration analyze of the piezoelectric nanobeams. They used DQM to study the effects of nonlocal parameter, temperature change and the external electric voltage on the nonlinear frequency of the piezoelectric nanobeams. Sobhy and Zenkour [22] discussed about magnetic field effect on thermomechanical buckling and vibration of viscoelastic sandwich nanobeams with CNT reinforced face sheets on a viscoelastic substrate. Also, Differential quadrature method for vibration analysis of electro-rheological sandwich plate with CNT reinforced nanocomposite facesheets subjected to electric field studied by Ghorbanpour Arani et al. [23]. Furthermore, in small-scales, nonlinear dynamic buckling analysis of embedded micro cylindrical shells reinforced with agglomerated CNTs using strain gradient theory was researched by Tohidi et al. [24].

With respect to developmental works on mechanical behavior analysis of nano and micro plates, it should be noted that none of the researches mentioned above have considered a coupled double-plate system. Herein, Murmu and Adhikari [25] analyzed vibration of nonlocal double nanoplate- system (NDNPS). Their study highlighted that the small-scale effects considerably influence the transverse vibration of NDNPS. Besides, they elucidated that the increase of the stiffness of coupling springs in the NDNPS reduces the small scale effects during the asynchronous modes of vibration. Also, buckling behavior of the NDNPS was investigated by Murmu et
al. [26] who showed that the nonlocal effects in the coupled system are higher within creasing values of the nonlocal parameter for the case of synchronous buckling modes than in the asynchronous buckling modes. Moreover, their analytical results indicated that the increase of the stiffness of the coupling springs in the double-GS-system reduces the nonlocal effects during the asynchronous modes of buckling. Exact solution for nonlocal vibration of double-orthotropic nanoplates embedded in elastic medium was reported by Pouresmaeeli et al. [27] who indicated that the frequency of double orthotropic nanoplates is always smaller than that of double isotropic nanoplates. The three papers [25-27] have considered the Winkler model for simulation of elastic medium between two nanoplates. In this simplified model, a proportional interaction between pressure and deflection of SLGSs is assumed, which is carried out in the form of discrete and independent vertical springs. Whereas, Pasternak suggested considering not only the normal stresses but also the transverse shear deformation and continuity among the spring elements, and its subsequent applications for developing the model for buckling analysis, which proved to be more accurate than the Winkler model. Recently, analysis of the coupled system of double layered graphene sheets (CS-DLGSs) embedded in a visco-Pasternak foundation is carried out by Ghorbanpour Arani et al. [28] who showed that the frequency ratio of the CS-DLGSs is more than the SLGS. To the best of our knowledge, none of the works in the literature have taken into account the nonlinear terms in the governing equations for a coupled system. This study aims to consider nonlinear terms for vibration analysis of a DPCMPS in which two microplates are connected by an enclosing Pasternak foundation.

None of the aforementioned studies [25-28] have considered smart coupled structures while these structures may be used in mechanical behavior control of coupled micro and nano structures. Recently, buckling analysis and smart control of SLGS using elastically coupled PVDF nanoplate using the nonlocal piezoelectricity were studied by Ghorbanpour et al. [29] who showed that the imposed external voltage is an effective controlling parameter for buckling of the SLGS. Moreover, their results indicated that the effect of external voltage becomes more prominent at higher nonlocal parameter and shear modulus. But paper [29] is linear analysis and just one of two plates is smart.

However, to date, no report has been found in the literature on the vibration of an elastically coupled DPCMPS. Motivated by these considerations, in order to improve optimum design of smart microstructure, the authors aim to study the electro-magneto nonlinear nonlocal vibration of an elastically coupled DPCMPS. Herein, the two PVDF microplates reinforced by agglomerated CNTs are coupled by an enclosing Pasternak foundation. Considering the nonlinear strain-displacement relations and charge equation, the nonlinear governing equations are derived using energy method and Hamilton’s principle. Hence, the DQM is presented to solve the nonlinear governing equations and estimate the frequency. In the present study, the influences of nonlocal parameter, temperature gradient, elastic medium constants, agglomeration and volume fraction of CNTs and magnetic field in polymer have been taken into account.

2. Formulation

2.1. Nonlocal Piezoelectricity

Based on the theory of nonlocal piezoelectricity, the stress tensor and the electric displacement at a reference point depend not only on the strain components and electric field components at the same position but also on all other points of the body. The nonlocal constitutive behavior for the piezoelectric material can be given as follows [21]:

$$\sigma''_{ij}(x) = \int a(|x-x'|, \tau) \sigma_{ij} dV(x'), \quad \forall x \in V$$  \hspace{1cm} (1)

$$D''_k = \int a(|x-x'|, \tau) D'_k dV(x'), \quad \forall x \in V$$ \hspace{1cm} (2)

where $\sigma''_{ij}$ and $\sigma'_{ij}$ are, respectively, the nonlocal stress tensor and local stress tensor, $D''_k$ and $D'_k$ are the components of the nonlocal and local electric displacement. $a(|x-x'|, \tau)$ is the nonlocal modulus, $|x-x'|$ is the Euclidean distance, and $\tau = e_o a / l$ is defined that $l$ is the external characteristic length, $e_o$ denotes a constant appropriate to each material, and $a$ is an internal characteristic length of the material. Consequently, $e_o a$ is a constant parameter which is obtained with molecular dynamics, experimental results, experimental studies, and molecular structure mechanics. The constitutive equation of the nonlocal elasticity can be written as [30]:

$$(1-\mu \nabla^2)\sigma''_{ij} = \sigma'_{ij},$$ \hspace{1cm} (3)

where the parameter $\mu = (e_o a)^2$ denotes the small scale effect on the response of structures in nano micro size and $\nabla^2$ is the Laplacian operator in the above equation. Similarly, Eq. (2) can be written as [20]:

$$(1-\mu \nabla^2)D''_k = D'_k,$$ \hspace{1cm} (4)

2.2. Classical Plate Theory

Based on the classical plate theory (CPT) which satisfies Kirchhoff assumption, the displacement field is expressed as [31]:
\[ u(x, y, z, t) = u_0(x, y, t) - \frac{\partial w_0}{\partial x}, \]
\[ v(x, y, z, t) = v_0(x, y, t) - \frac{\partial w_0}{\partial y}, \]
\[ w(x, y, z, t) = w_0(x, y, t), \]

where \( u, v, w \) denote the total displacements of a point along the \((x, y, z)\) coordinates and \((u_0, v_0, w_0)\) are the displacements of points on the mid-plane. The von-Kármán nonlinear strains associated with the above displacement field can be expressed in the following form \([32]\):

\[
\begin{align*}
\varepsilon_{xx} &= \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2, \\
\varepsilon_{yy} &= \frac{\partial v}{\partial y} + \frac{1}{2} \left( \frac{\partial w}{\partial y} \right)^2, \\
\varepsilon_{zz} &= \frac{\partial w}{\partial z} \left( \frac{\partial w}{\partial x} \right), \\
\varepsilon_{xy} &= \frac{1}{2} \left( \frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} \right), \\
\varepsilon_{xz} &= \frac{1}{2} \left( \frac{\partial w}{\partial x} + \frac{\partial w}{\partial z} \right), \\
\varepsilon_{yz} &= \frac{1}{2} \left( \frac{\partial w}{\partial y} + \frac{\partial w}{\partial z} \right),
\end{align*}
\]

\[ \varepsilon_{xy} = \varepsilon_{yx} = \frac{\partial w}{\partial y} + \frac{\partial w}{\partial x}, \]

On the basis of the CPT, shear strains \( \varepsilon_{xy}, \varepsilon_{yz} \) are considered negligible. Hence, the strain equations in terms of the mid-plane displacements are derived by substituting the Eq. (5) into the Eq. (6) as follows:

\[
\begin{align*}
\varepsilon_{xx} &= \varepsilon_{xx}^0 + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2, \\
\varepsilon_{yy} &= \varepsilon_{yy}^0 + \frac{1}{2} \left( \frac{\partial w}{\partial y} \right)^2, \\
\gamma_{xy} &= \gamma_{xy}^0 + \frac{1}{2} \left( \frac{\partial w}{\partial y} + \frac{\partial w}{\partial x} \right), \\
\gamma_{xz} &= \gamma_{xz}^0 + \frac{1}{2} \left( \frac{\partial w}{\partial z} + \frac{\partial w}{\partial x} \right), \\
\gamma_{yz} &= \gamma_{yz}^0 + \frac{1}{2} \left( \frac{\partial w}{\partial z} + \frac{\partial w}{\partial y} \right),
\end{align*}
\]

The strain components \( \varepsilon_{xx}, \varepsilon_{yy} \) and \( \gamma_{xy} \) at an arbitrary point of the sheet are related to the mid-surface strains and curvatures tensor as follows:

\[
\begin{align*}
\varepsilon_{xx} &= \varepsilon_{xx}^0 + \varepsilon_{xx}^1, \\
\varepsilon_{yy} &= \varepsilon_{yy}^0 + \varepsilon_{yy}^1, \\
\gamma_{xy} &= \gamma_{xy}^0 + \gamma_{xy}^1,
\end{align*}
\]

where \( \varepsilon_{xx}^0, \varepsilon_{yy}^0, \gamma_{xy}^0 \) are components of the membrane strains (mid-surface strains) tensor and \( \varepsilon_{xx}^1, \varepsilon_{yy}^1, \gamma_{xy}^1 \) are components of the bending strain (curvature) tensor.

### 2.3. Modeling of the Problem

An elastically coupled DPCMPS having the length \( l \), the width \( b \) and the thickness \( h \), assuming that \( h \ll l, b \) \([32]\), is shown in Fig. 1.

The origin of the Cartesian coordinate system is considered at one corner of the middle surface of the microplate. The \( x, y \), and \( z \) axes are taken along the length, width, and thickness of the microplates, respectively. The two microplates are made from PVDF and reinforced by CNTs in \( x \)-direction so that both microplates are identical. The DPCMPS is subject to uniform temperature change and polarized in \( x \)-direction. The two microplates are coupled by an elastic medium which is simulated by the Pasternak foundation. As is well known, this foundation model is characterized by two parameters: the Winkler constant \( k_w \) and shear constant \( k_p \).

#### 2.4. Constitutive Equations for Piezoelectric Materials

In a piezoelectric material, application of an electric field to it will cause a strain proportional to the mechanical field strength, and vice versa. According to a piezoelectric microplate under electro-thermal loads, constitutive equations can be represented as \([33]\):

\[
(1 - \mu N^2) \begin{bmatrix}
\sigma_{xx}^0 \\
\sigma_{yy}^0 \\
\sigma_{xy}^0 \\
\sigma_{xz}^0 \\
\sigma_{yz}^0
\end{bmatrix} = \begin{bmatrix}
C_{11} & C_{12} & 0 \\
C_{12} & C_{22} & 0 \\
0 & 0 & C_{66}
\end{bmatrix} \begin{bmatrix}
\varepsilon_{xx} \\
\varepsilon_{yy} \\
\gamma_{xy}
\end{bmatrix} + \begin{bmatrix}
\alpha_x \\
\alpha_y \\
\alpha_t
\end{bmatrix} \Delta T
\]

\[
(1 - \mu N^2) \begin{bmatrix}
D_{xx} \\
D_{yy} \\
D_{xy}
\end{bmatrix} = \begin{bmatrix}
e_{11} & e_{12} & 0 \\
e_{12} & e_{22} & 0 \\
e_{26} & 0 & e_{66}
\end{bmatrix} \begin{bmatrix}
\varepsilon_{xx} \\
\varepsilon_{yy} \\
\gamma_{xy}
\end{bmatrix} - \begin{bmatrix}
\alpha_x \\
\alpha_y \\
\alpha_t
\end{bmatrix} \Delta T
\]

where \( e_{ij}, \varepsilon_{ij} (i, j = 1, \ldots, 6), \alpha (k=x,y) \) and \( \Delta T \) are piezoelectric constants, dielectric constants, thermal expansion coefficients, and temperature gradient, respectively. \( C_{ij} \) are components of stiffness tensor. Electric
field tensor $E$ can be written in term of electric potential $\phi$ as [34]:
$$E = -\nabla \phi. \quad (11)$$

2.5. Mori-Tanaka Approach

In this section, the effective modulus of the composite shell reinforced by CNTs is developed. Different methods are available to estimate the overall properties of a composite [1]. Due to its simplicity and accuracy even at high volume fractions of the inclusions, the Mori-Tanaka method [1] is employed in this section. To begin with, the CNTs are assumed to be aligned and straight with the dispersion of uniform in the polymer. The matrix is assumed to be elastic and isotropic, with Young’s modulus $E_m$ and the Poisson’s ratio $\nu_m$. The constitutive relations for a layer of the composite with the principal axes parallel to the $r$, $\theta$ and $z$ directions are [1]:

$$\begin{bmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{33} \\
\sigma_{23} \\
\sigma_{13} \\
\sigma_{12}
\end{bmatrix} = 
\begin{bmatrix}
k + m & l & k - m & 0 & 0 & 0 \\
l & n & l & 0 & 0 & 0 \\
k - m & l & k + m & 0 & 0 & 0 \\
0 & 0 & 0 & p & 0 & 0 \\
0 & 0 & 0 & 0 & m & 0 \\
0 & 0 & 0 & 0 & 0 & p
\end{bmatrix}
\begin{bmatrix}
\varepsilon_{11} \\
\varepsilon_{22} \\
\varepsilon_{33} \\
\gamma_{23} \\
\gamma_{13} \\
\gamma_{12}
\end{bmatrix} \quad (12)$$

where $\sigma_{ij}$, $\varepsilon_{ij}$, $\gamma_{ij}$, $k$, $m$, $n$, $l$ and $p$ are the stress components, the strain components, and the stiffness coefficients, respectively. According to the Mori-Tanaka method, the stiffness coefficients are given by [1]:

$$k = \frac{E_m [E_m c_{nn} + 2k (1 + \nu_m) (1 + c_k (1 - 2\nu_m))]}{2(1 + \nu_m) [E_m (1 + c_k - 2\nu_m) + 2c_k, (1 - \nu_m - 2\nu_m)]}$$

$$l = \frac{E_m [c_r v_r (E_m + 2k (1 + \nu_m) + 2c_k, (1 - \nu_m - 2\nu_m))]}{(1 + \nu_m) [E_m (1 + c_k - 2\nu_m) + 2c_k, (1 - \nu_m - 2\nu_m)]}$$

$$n = \frac{E_m [c_r (1 + c_k - c_r v_r) + 2c_k, (k, n, l, l')(1 + \nu_m) (1 - 2\nu_m)]}{(1 + \nu_m) [E_m (1 + c_k - 2\nu_m) + 2c_k, (1 - \nu_m - 2\nu_m) (1 - 2\nu_m)]}$$

$$\begin{aligned}
&+ E_m [2c_k, (1 - \nu_m + c_k, (1 + c_k - 2\nu_m) - 4c_k, l + v_r)] \\
&+ E_m (1 + c_k - 2\nu_m) + 2c_k, (1 - \nu_m - 2\nu_m)
\end{aligned}$$

$$p = \frac{E_m [c_r v_r + 2p_r (1 + v_r) (1 + c_k)]}{2(1 + \nu_m) [E_m (1 + c_k) + 2c_m, p_r, (1 + v_m)]}$$

$$m = \frac{E_m [c_r v_r + 2m_r (1 + v_r) (3c_k - 4c_m)]}{2(1 + \nu_m) [E_m (c_m + 4c_r, (1 + v_m) + 2c_m, m, (3 - v_m - 4v_m)]}$$

where $C_m$ and $C_r$ are the volume fractions of the matrix and the CNTs respectively and $k$, $l$, $n$, $p$, $m$ are the Hills elastic modulus for the CNTs [1]. The experimental results show that the most of CNTs are bent and centralized in one area of the polymer. These regions with concentrated CNTs are assumed in this section to have spherical shapes and are considered as “inclusions” with different elastic properties from the surrounding material. The total volume $V$ of CNTs can be divided into the following two parts [1]:

$$V = V_{inclusion} + V_{out} \quad (14)$$

where $V_{inclusion}$ and $V_{out}$ are the volumes of CNTs dispersed in the inclusions, concentrated regions and in the matrix, respectively. Introduce two parameters $\xi$ and $\zeta$ describe the agglomeration of CNTs:

$$\xi = \frac{V_{inclusion}}{V} \quad (15)$$

$$\zeta = \frac{V_{out}}{V} \quad (16)$$

However, the average volume fraction $C$ of CNTs in the composite is:

$$C = \frac{V_r}{V} \quad (17)$$

Assume that all the orientations of the CNTs are completely random. Hence, the effective bulk modulus ($K$) and effective shear modulus ($G$) may be written as:

$$K = K_{out} + \frac{\chi_0 (\delta - 3K_m C_r)}{3(\xi - C_r C_m C_r) + C_r C_m C_r} \quad (18)$$

$$G = G_{out} + \frac{\chi_0 (\delta - 3G_m C_r)}{2(\xi - C_r C_m C_r) + C_r C_m C_r} \quad (19)$$

where:

$$K_m = K_m + \frac{(\delta - 3K_m C_r)}{3(\xi - C_r C_m C_r) + C_r C_m C_r} \quad (20)$$

$$K_{out} = K_m + \frac{C_r (\delta - 3K_m C_r) (1 - \zeta)}{3[1 - \xi - C_r (1 - \zeta) + C_r C_m (1 - \zeta)]} \quad (21)$$

$$G_m = G_m + \frac{(\eta - 3G_m C_r) \beta C_r}{2(\xi - C_r C_m C_r) + C_r C_m C_r} \quad (22)$$

$$G_{out} = G_m + \frac{C_r (\delta - 3G_m C_r) (1 - \zeta)}{2[1 - \xi - C_r (1 - \zeta) + C_r C_m (1 - \zeta)]} \quad (23)$$

where $\chi_0$, $\beta$, $\delta$, $\eta$, $\gamma$ may be calculated as:

$$\chi_0 = \frac{3(K_m + G_m) + \kappa - \lambda}{3(K_m + G_m)} \quad (24)$$

$$\beta = \frac{3(k_1 + G_m)}{4G_m + 2k_1 + l + 4G_n + \frac{1}{4}[G_m (3K_m + G_m) + G_n (3K_n + 7G_n)]} \quad (25)$$

$$\delta = \frac{1}{3} \left[ n_1 + 2l_1 + \frac{(2k_1 - l_1) (3K_m + 2G_m - l_1)}{k_1 + G_m} \right] \quad (26)$$
where \(K_n\) and \(G_m\) are the bulk and shear moduli of the matrix which can be written as:

\[
K_n = \frac{E_n}{3(1-2\nu_n)} \quad \text{(28)}
\]

\[
G_m = \frac{E_m}{2(1+\nu_m)} \quad \text{(29)}
\]

Furthermore, \(\beta\) and \(\alpha\) can be obtained from:

\[
\alpha = \frac{(1+\nu_{out})}{3(1-\nu_{out})} \quad \text{(30)}
\]

\[
\beta = \frac{2(4-5\nu_{out})}{15(1-\nu_{out})} \quad \text{(31)}
\]

\[
\nu_{out} = \frac{3K_{out}-2G_{out}}{6K_{out}+2G_{out}} \quad \text{(32)}
\]

Finally, the elastic modulus \(E\) and Poisson’s ratio \(\nu\) can be calculated as:

\[
E = \frac{9KG}{3K+G} \quad \text{(33)}
\]

\[
\nu = \frac{3K-2G}{6K+2G} \quad \text{(34)}
\]

### 2.6. Equations of Motion

The governing differential equations of motion are derived using Hamilton’s principle which is given as [35]:

\[
\int_0^t (\delta U + \delta V - \delta K) dt = 0 \quad \text{(35)}
\]

where \(\delta U\) is the virtual strain energy which is obtained by the following relation [36]:

\[
\delta U = \frac{1}{2} \iiint [\sigma_{ij}e_{ij} - D_e E_i] dV \quad \text{(36)}
\]

\(\delta K\) is the virtual kinetic energy and is defined as follow [37]:

\[
\delta K = \frac{1}{2} \iiint \left[ (\dot{u}^2) + (\dot{v})^2 + (\dot{w})^2 \right] dV \quad \text{(37)}
\]

Also, \(\delta V\) is the virtual work done by external applied forces and is obtained by:

\[
\delta U = \frac{1}{2} \iiint [q + q_m] dV \quad \text{(38)}
\]

in which \(q_m\) can be written as:

\[
q_m = \eta H_i \frac{\partial^2 w}{\partial x^2} \quad \text{(39)}
\]

where \(\eta\) is the magnetic permeability; \(\nabla\) is the gradient operator; \(H_i\) is the magnetic field. Furthermore, \(q_e\) can be written as:

\[
q_e = k_w w - k_g \nabla^2 w \quad \text{(40)}
\]

where \(k_w\) and \(k_g\) are spring and shear constants of elastic medium, respectively.

The motion equations can be derived using Eq. (35) as follows:

\[
\frac{\partial N_w}{\partial x} + \frac{\partial N_{ww}}{\partial y} = m_0 \frac{\partial^2 w_0}{\partial x^2} \quad \text{(41)}
\]

\[
\frac{\partial N_{ww}}{\partial x} + \frac{\partial N_{ww}}{\partial y} = m_0 \frac{\partial^2 v_0}{\partial y^2} \quad \text{(41)}
\]

\[
\frac{\partial^2 M_{xx}}{\partial x^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_{yy}}{\partial y^2} \quad \text{(41)}
\]

For \(\{M_{xx}, M_{xy}, M_{yy}\}\) and the moment resultants \(\{N_{xx}, N_{xy}, N_{yy}\}\) of the plate can be define as:

\[
\{(N_{xx}, N_{xy}, N_{yy}), (M_{xx}, M_{xy}, M_{yy})\} \quad \text{(43)}
\]

Charge equation for coupling electrical and mechanical fields is:

\[
\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + D_z = 0 \quad \text{(44)}
\]

In this study, transverse vibration is investigated (i.e. \(w_0 = v_0 = 0\)).

### 3. DQ Method

As can be seen, the coupled governing equations contain nonlinear terms and should be solved using a numerical method such as DQ. In this method, the differential equations are changed into a first-order algebraic equation by employing appropriate weighting coefficients. Weighting coefficients do not relate to any special problem and only depend on the grid spacing. For the implementation of the DQ approximation, consider a function \(f(\xi, \eta)\) which has the field on a rectangular domain \((0 \leq \xi \leq 1\) and \(0 \leq \eta \leq 1)\) with \(n_\xi \times n_\eta\)
grid points along \( x \) and \( y \) axes. According to DQ method, the \( r \)th derivative of a function \( f(x, y) \) can be defined as [35]:

\[
\frac{\partial^r f (\zeta, \eta)}{\partial \zeta^r} \bigg|_{\zeta=\zeta_i, \eta=\eta_j} = \sum_{n=1}^{n_r} C^{r}_{ij} f (\zeta_m, \eta_i) \bigg|_{\zeta=\zeta_j, \eta=\eta_j}
\]

(45)

where \( C^{r}_{ij} \) are weighting coefficients and defined as:

\[
C^{r}_{ij} = \begin{cases} 
M(\zeta_j) & \text{for } i \neq j \\
- \sum_{m \neq i}^{n_r} C^{(r)}_{ij} & \text{for } i = j
\end{cases}
\]

(46)

where \( M(\zeta_j) \) is Lagrangian operators which can be presented as:

\[
M(\zeta_j) = \prod_{i \neq j}^{n_r} (\zeta_j - \zeta_i).
\]

(47)

The weighting coefficients for the second, third and fourth derivatives are defined as:

\[
C^{(2)}_{ij} = \sum_{k=1}^{n_r} C^{(1)}_{ik} C^{(1)}_{kj},
\]

\[
C^{(3)}_{ij} = \sum_{k=1}^{n_r} C^{(2)}_{ik} C^{(1)}_{kj} = \sum_{k=1}^{n_r} C^{(2)}_{ik} C^{(1)}_{kj},
\]

\[
C^{(4)}_{ij} = \sum_{k=1}^{n_r} C^{(3)}_{ik} C^{(1)}_{kj} = \sum_{k=1}^{n_r} C^{(3)}_{ik} C^{(1)}_{kj}.
\]

(48)

In a similar method, the weighting coefficients for \( y \)-direction can be obtained. The coordinates of grid points are chosen as:

\[
\zeta_i = \frac{1}{2} \left[ 1 - \cos \left( \frac{\pi (i-1)}{n_x} \right) \right],
\]

\[
\eta_j = \frac{1}{2} \left[ 1 - \cos \left( \frac{\pi (j-1)}{n_y} \right) \right].
\]

(49)

In order to carry out the eigenvalue analysis, the domain and boundary points are separated and in vector forms, they are denoted as \( \{d\} \) and \( \{b\} \), respectively. Hence, the discretized form of the motion equations together with the boundary conditions can be expressed in matrix form as:

\[
\begin{pmatrix}
[K_i + K_{NL}] + \Omega_i^2 [M] \\
[K] + \Omega_i^2 [N]
\end{pmatrix}
\begin{pmatrix}
d \\
b
\end{pmatrix} = 0,
\]

(50)

in which \([M], [K_i] \) and \([K_{NL}] \) are the mass matrix, linear stiffness matrix, and nonlinear stiffness matrix. This nonlinear equation can now be solved using a direct iterative process as follows:

- First, nonlinearity is ignored by taking \([K_{NL}]=0 \) to solve the eigenvalue problem expressed in equation (50). This yields the linear eigenvalue \( (\Omega_i) \) and associated eigenvector. The eigenvector is then scaled up so that the maximum transverse displacement of the microplate is equal to the maximum eigenvector, i.e. the given vibration amplitude \( W_{max} \).
- Using linear eigenvector, \([K_{NL}] \) could be evaluated. Eigenvalue problem is then solved by substituting \([K_{NL}] \) into equation (50). This would give the nonlinear eigenvalue \( (\Omega_i) \) and the new eigenvector.
- The new nonlinear eigenvector is scaled up again and the above procedure is repeated iteratively until the frequency values from the two subsequent iterations ‘r’ and ‘r+1’ satisfy the prescribed convergence criteria [38] as:

\[
\frac{\omega^{r+1} - \omega^{r}}{\omega^{r}} < \varepsilon_0,
\]

(51)

where \( \varepsilon_0 \) is a small value number and in the present analysis it is taken to be 0.1%.

4. Numerical Results and Discussion

Mechanical, thermal and electrical properties of PVDF matrix and CNT reinforcement are chosen from Ref. [9]. The final converged solution using the numerical procedure above is illustrated as the influences of the elastic medium, nonlocal parameter, volume percent of CNT, CNTs agglomeration and temperature change on the frequency of the structure.

Since no reference to such a work is found to-date in the literature, its validation is not possible. However, the present work could be partially validated based on a simplified analysis suggested by Shen et al. [14] on thermal nonlinear vibration of the SLGS for which the coupled plate and volume percent of CNTs in the polymer were ignored. For this purpose, a SLGS with \( \varepsilon=0 \), \( T=300 K \), \( h=9.496 \) nm, \( b=4.877 \) nm, \( h=0.145 \) nm, \( \rho=5624 \) kg/m\(^3\), \( k_w=k_h=0 \) and \( e_{oa}=0.67 \) nm is considered. Table 1 illustrates the result of validation exercise by showing nonlinear-to-linear frequency for different dimensionless amplitude \( (w/h) \) and temperature. As can be seen, the results obtained are in good agreement with those expressed in [14].

Fig. 2 demonstrates the effects of the nonlocal parameter on the frequency versus the orientation angle of CNTs.

<table>
<thead>
<tr>
<th>Temperature</th>
<th>Ref.</th>
<th>( \Omega_{nl}/\Omega_c )</th>
</tr>
</thead>
<tbody>
<tr>
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<td></td>
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</table>

Table 1. Comparing dimensionless nonlinear frequency obtained in the present study and those of Shen et al. [14].
As can be seen, the frequency of system decreases with considering size effects. This is due to the fact that the considering of nonlocal parameter decreases the interaction force between microplate atoms, and that leads to a softer structure.

The effect of volume percent of CNTs on the frequency versus the orientation angle of CNTs is shown in Fig. 3. It is clear that the frequency increases with increasing the volume percent of CNTs and the influence of volume percent of CNTs on the frequency becomes more prominent at the middle angle. It is because with increasing the volume percent of CNTs, the stiffness of structure increases.

Fig. 4 demonstrates agglomeration effects on the frequency versus orientation angle of CNTs. As can be seen, considering agglomeration effects leads to lower frequency since the stability of system decreases.

Fig. 5 illustrates the influence of thermal gradient ($\Delta T$) on the frequency versus the orientation angle of CNTs. It is evident that an increase in temperature change does not considerable effect on the frequency.

Fig. 6 shows the effect of temperature variations on the results. It is seen by increasing the temperature difference, the frequency enhances due to change in mechanical properties of the structure.

The effect of elastic medium on the frequency of structure is shown in Fig. 7. It can be found that considering elastic medium leads to higher frequency. In addition, considering Pasternak medium predicts the higher frequency with respect to Winkler medium. It is because in Pasternak medium, the normal and shear constant are considered.
The effects of magnetic field on the frequency of structure is shown in Fig. 9. It can be found that increasing the magnetic field, the frequency increases. It is because increasing the magnetic field leads to higher stiffness.

5. Conclusions

Vibration response of piezoelectric nano/micro composites has applications in designing many NEMS/MEMS devices such as hydraulic sensors and actuators. In the present study, electro-magneto nonlinear vibration of a double-piezoelectric composite microplate made of PVDF reinforced by CNTs is investigated considering agglomeration effects. The internal elastic medium between two microplates is simulated as Pasternak foundation. Considering charge equation, the nonlinear motion equations are derived based on nonlocal piezoelectricity theory. The DQM is applied to obtain the nonlinear frequency ratio of the DPCMPS so that the effects of the small scale coefficient, stiffness of the internal elastic medium, the vol-
ume fraction and orientation angle of the CNTs reinforcement, temperature change and agglomeration are discussed. The results of this study are validated by Shen et al. [14]. The results indicate that with increasing geometrical aspect ratio, the effect of coupling elastic medium between two piezoelectric composite microplates decreases. Furthermore, the effects of small scale parameter and volume percent lead to the higher frequency. It is also worth mentioning that the frequency of structure considering the agglomeration of CNTs becomes lower.

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References

to electric field. *Composite Structures* 2017; 180: 211-20.


