A novel method for considering interlayer effects between graphene nanoribbons and elastic medium in free vibration analysis

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ABSTRACT

A complete investigation on the free vibration of bilayer graphene nanoribbons (BLGNRs) modeled as sandwich beams taking into account tensile-compressive and shear effects of van der Waals (vdWs) interactions between adjacent graphene nanoribbons (GNRs) as well as between GNRs and polymer matrix is performed in this research. In this modeling, nanoribbon layers play role of sandwich beam layers and are modeled based upon Euler-Bernoulli theory. To consider effects of vdWs interactions between adjacent GNRs as well as between GNRs and polymer matrix, their equivalent tensile-compressive and shear moduli are considered and utilized in derivation of governing equations instead of employing conventional Winkler and Pasternak effects for elastic medium. The governing equations of motion are derived by considering the assumptions and employing sandwich beam theory, and natural frequencies are obtained by implementing harmonic differential quadrature method (HDQM). A detailed study is performed to examine the influences of the tensile-compressive and shear effects of vdWs interactions between adjacent GNRs as well as between GNRs and polymer matrix on the free vibration of BLGNRs.

Keywords: Elastic medium, Sandwich theory, Tensile-compressive effects, Shear effects, Euler-Bernoulli theory

1. Introduction

Graphenes can be synthesized either single-layer or multi-layer. Layers of multi-layer graphenes are located next to each other by weak interactions which known as vdWs interactions. These weak interactions change the mechanical and electrical properties of multi-layer graphenes [1] where it can be attributed to the tensile-compressive and shear effects of vdWs interactions. With an overview of the references studying on the mechanical behavior of multilayer graphene sheets (MLGSs) and multi-layer graphene nanoribbons (MLGNRs), it is found that they can be classified in two categories. In the first category, only tensile-compressive effects of van der Waals (vdWs) interactions between two graphene layers are considered on the mechanical behavior of MLGSs [2-11] or MLGNRs [12-16]. For example, Ansari et. al [17] have investigated effects of number of graphene layers and nonlocal parameter by employing Reissner-Mindlin plate theory on the free vibration of MLGSs. In the second category, a handful of researchers have considered only shear effects of vdWs interactions between two graphene layers on the mechanical behavior of MLGNRs [18, 19]. For example, Liu et. al [20] have investigated the bending of cantilever bilayer and trilayer graphene nanoribbons incorporating the interlayer shear by employing Newmark's composite beam theory. They show that considering of in-plane displacement of nanoribbons has significant effect on the bending of multilayer graphene nanoribbons. This literature survey shows that researchers have only considered one of the vdWs interactions effects between GNR layers, the tensile-compressive effect or the shear one. Therefore, there is no literature

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investigating the tensile-compressive and shear effects of vdWs interactions between adjacent graphene sheets (GSs) or GNRs, simultaneously.

It is known that in macro dimensions the tensile-compressive and shear effects of elastic medium are modeled by Winkler and Pasternak terms defined in terms of transverse displacement. This approach is also used in nano dimensions for embedded GSs or GNRs in elastic medium since vdWs interactions are observed not only between adjacent graphene layers but also between elastic medium and graphene layers [10, 11, 15]. All of these studies have considered only transverse displacement of graphene layers. In other words, the in-plane displacement of graphene layers has not been considered while it has been mentioned in previous paragraph that considering of the in-plane displacement of graphene layers has significant effect on the mechanical behavior of MLGSs and MLGNRs.

From the above consideration, two questions arise. First, what will the tensile-compressive and shear effects of vdWs interactions between adjacent graphene layers be on free vibration of GNRs when they are simultaneously considered? Second, what will the shear effect of elastic medium be on free vibration of GNRs if the in-plane displacement of GNRs is considered? To cover the questions, a GNR is modeled based on the sandwich beam theory and tensile-compressive and shear effects of vdWs interactions between adjacent GNRs as well as elastic medium and GNRs are modeled as equivalent tensile-compressive and shear moduli in equation of motion. The governing equations of motion are derived by using Hamilton’s principle and solved numerically by HDQM. Then natural frequencies are obtained for clamped boundary condition. Some comparison studies are performed to show the accuracy of formulation and solution procedure. The effects of equivalent tensile-compressive and shear moduli of vdWs interactions between elastic medium and GNRs on the first five natural frequencies of BLGNR are investigated. Finally, the effects of equivalent tensile-compressive and shear moduli of vdWs interaction between elastic medium and GNRs on the natural frequencies of BLGNR are numerically compared with those of vdWs interactions between adjacent GNRs.

2. Problem Formulation

Consider a BLGNR with the surrounding elastic medium in a continuum model as shown in Fig. 1. For better understanding of the study procedure, the flowchart diagram is presented for step by step understanding as Fig. 2. The model consists of five layers: two GNRs, a low density core connecting GNRs to each other (vdWs interactions), and two elastic mediums. Elastic mediums are bonded to the GNRs on the one side and connected to a fix layer on the other side. All five layers are firmly bonded together and vdWs interactions inertia is not notable. It is important to be noted that considered elastic mediums are a type of vdWs interactions which connect GNRs to a polymer matrix such as Polyethylene [16]. The Cartesian coordinate system is used and the origin is located at the left-hand side of BLGNR in the middle of core thickness. The x and z coordinates of axes are taken along the length and thickness of BLGNR, respectively. Here \( L, b, h_f, h_c \) and \( h_e \) denote length, width, thickness of GNR layers, thickness of core, and thickness of elastic mediums, respectively. The displacement components along \( x \) and \( z \) are illustrated by \( u \) and \( w \), respectively.

![Fig. 1. Geometry and coordinate system of BLGNR embedded in an elastic medium.](image)

The vdWs interactions of elastic mediums are modeled in a way that they can withstand tensile-compressive and shear forces simultaneously. The vdWs interactions between GNR and polymer matrix can be modeled stronger or weaker than vdWs interactions between GNRs with each other. The GNRs are modeled based on Euler-Bernoulli beam theory. According to the theory the displacement field of the upper face \( u_f \) and \( w_f \) and the lower face \( u_b \) and \( w_b \) are defined as follow [18]:

\[
\begin{align*}
    u_f(x,z,t) &= u_f(x,t) - \left( z - \frac{h_f + h_c}{2} \right) \frac{\partial w_f}{\partial x} \quad (1) \\
    w_f(x,z,t) &= w_f(x,t) \quad (2) \\
    u_b(x,z,t) &= u_b(x,t) - \left( z + \frac{h_f + h_c}{2} \right) \frac{\partial w_b}{\partial x} \quad (3)
\end{align*}
\]
in which $u_1$, $w_1$, and $u_2$, $w_2$ denote the displacements of an arbitrary point on mid-axis of the top and bottom layers, respectively, $h_f$ and $h_c$ are the thickness of the nanoribbon layers and the core, respectively, and $z$ is measured from the $h_c/2$. The strain components of GNRs can be calculated as [18]:

$$w_b(x,z,t) = w_2(x,t)$$ (4)

As a general method, the elastic mediums are modeled as Winkler or Pasternak foundation [10, 11, 21, 22]. Pasternak foundation considers only normal pressure, while Winkler foundation describes not only normal pressure but also transverse shear stress. On the contrary to the conventional, the authors present a new model to consider the tensile-compressive and shear effects of the elastic medium for the first time. The modeling is based on the sandwich theory and it is assumed that elastic medium has specific thickness and longitudinal and transverse displacements of the elastic medium are varied linearly through the elastic medium thickness. Since the elastic medium is vdW's interactions between fixed matrix and GNRs, the displacement equations of the elastic medium can be obtained as below:

$$u'_b = \left(\frac{1}{2} - \frac{z_f'}{h_c}\right)\left(u_1 - \frac{h_f}{h_c} \frac{\partial w_1}{\partial x}\right)$$ (9)

$$w'_b = \left(\frac{1}{2} - \frac{z_f'}{h_c}\right)w_1$$ (10)

$$u''_b = \left(\frac{1}{2} - \frac{z''_b}{h_c}\right)\left(u_2 + \frac{h_f}{h_c} \frac{\partial w_2}{\partial x}\right)$$ (11)

$$w''_b = \left(\frac{1}{2} + \frac{z''_b}{h_c}\right)w_2$$ (12)

in which $u'_b$, $w'_b$ and $u''_b$, $w''_b$ denote the displacements of the top and bottom elastic mediums, $h_c$ is the elastic medium thickness, and $z'_b$ and $z''_b$ are measured from the $h_c/2$. Since the core and elastic mediums are considered not to resist in-plane loading, their longitudinal strains are insignificant. While bending and shear strains of the core ($\varepsilon^e_{xx}$ and $\gamma^e_{xz}$) and elastic mediums ($\varepsilon^{e,t}_{xx}$, $\varepsilon^{e,b}_{xx}$, $\gamma^{e,t}_{xz}$, and $\gamma^{e,b}_{xz}$) are significant and expressed as follow:

$$\varepsilon^e_{xx} = \frac{\partial w_c}{\partial z} = \frac{w_1 - w_2}{h_c}$$ (13)

$$\gamma^e_{xz} = \frac{\partial u_c}{\partial z} + \frac{\partial w_c}{\partial x} = \left[\frac{u_1 - u_2}{2} + \frac{h_f}{4} \frac{\partial w_1}{\partial x} + \frac{\partial w_2}{\partial x}\right] \frac{2z}{h_c} + \left[u_1 + u_2 + \frac{h_f}{2} \frac{\partial w_1}{\partial x} + \frac{\partial w_2}{\partial x}\right] \frac{h_f}{h_c}$$ (14)

$$\varepsilon^{e,t}_{xx} = \frac{\partial w^t_{e}}{\partial z^t_{e}} = -\frac{w_1}{h_c}$$ (15)
\[ e^{e.b}_{xx} = \frac{\partial w^b}{\partial z^b_e} \frac{w_2}{h_e} \]

\[ \gamma^{e.t}_{xx} = \frac{\partial u^t}{\partial z^t_e} + \frac{\partial w^t}{\partial x} = \left( \frac{h_f \partial w_1 - u_1}{h_e} + \left( \frac{1}{2} - \frac{z^t_e}{h_e} \right) \frac{\partial w_1}{\partial x} \right) \]

\[ \gamma^{e.b}_{xx} = \frac{\partial u^b}{\partial z^b_e} + \frac{\partial w^b}{\partial x} = \left( \frac{h_f \partial w_2 + u_2}{h_e} + \left( \frac{1}{2} + \frac{z^b_e}{h_e} \right) \frac{\partial w_2}{\partial x} \right) \]

Now the stress-strain relations for the core, two faces, and two elastic mediums can be obtained as

\[ \sigma^k_{xx} = E_f e^{e.k}_{xx} ; \quad k = t, b \]

\[ \sigma^c_{xx} = E_c e^{c.x}_{xx} \]

\[ \sigma^{e,k}_{xx} = E_e e^{e,k}_{xx} ; \quad k = t, b \]

where \( E_f, E_c, E_e \) and \( G_f \) are elastic modulus of a single layer of GNR, tensile-compressive modulus of the core, shear modulus of the core, tensile-compressive modulus of the elastic medium, and shear modulus of the elastic medium, respectively. \( \sigma_{xx} \) and \( \sigma_{zz} \) are the normal stresses in the x and z directions, respectively, and \( \tau_{xz} \) is the shear stress in the xz plane.

By employing the Hamilton’s principle (equation (22)), where \( U \) and \( T \) are potential energy and kinetic energy and \( t \) is as time. \( \delta u \) and \( \delta T \) are defined as equation (23-26).

\[ \int_{t_1}^{t_2} (\delta u - \delta t) \, dt = 0 \]

\[ \delta u_1: -E_f A_f \frac{\delta^2 u_1}{\delta x^2} + \left( \frac{G_c A_c}{h_c} + \frac{G_e A_e}{h_e} \right) u_1 - \left( \frac{G_c A_c}{h_c^2} \right) u_2 + \left( \frac{G_e A_e}{h_e^2} \right) \frac{\partial w_2}{\partial x} = 0 \]

\[ \delta u_2: -E_f A_f \frac{\delta^2 u_2}{\delta x^2} - \left( \frac{G_c A_c}{h_c^2} \right) u_1 + \left( \frac{G_c A_c}{h_c} + \frac{G_e A_e}{h_e} \right) u_2 - \left( \frac{G_e A_e}{2h_e^2} \right) \frac{\partial w_1}{\partial x} = 0 \]

where \( A_f, A_c, A_e \) are cross sectional area of the faces, the core and the elastic mediums, respectively. Also \( l_f, l_c \) and \( l_e \) are second moment of area of the faces, the core and the elastic mediums, respectively and \( \rho \) is the density of graphene nanoribbons. The developed coupled equations (equations (23)-(26)) are the governing equations of motion in which their couplings are due to the tensile-compressive and shear effects of vdWs interactions between carbon-carbon atoms as well as vdWs interactions between nanoribbons and polymer matrix. According to Hamilton’s principle the boundary conditions are also generated as follow:
\[
\delta u_1 : E_f A_f \frac{\partial u_1}{\partial x} = 0 \\
\delta u_2 : E_f A_f \frac{\partial u_2}{\partial x} = 0
\] (27)

\[
\delta w_1 : - E_f \frac{\partial^3 w_1}{\partial x^3} + \left( \frac{G_c A_c (h_f + h_e)}{2h_e^2} - \frac{G_c A_e (h_f + h_e)}{2h_e^2} \right) u_1 - \left( \frac{G_c A_c (h_f + h_e)}{2h_e^2} \right) u_2 \\
+ \left( \frac{G_c A_c (h_f + h_e)}{4h_e^2} + \frac{G_c l_c}{h_e^2} + \frac{G_e A_e (h_f + h_e)^2}{4h_e^2} \right) \frac{\partial w_1}{\partial x} + \left( \frac{G_c A_c (h_f + h_e)}{4h_e^2} - \frac{G_c l_c}{h_e^2} \right) \frac{\partial w_2}{\partial x} = 0
\] (29)

\[
\delta w_2 : - E_f \frac{\partial^3 w_2}{\partial x^3} + \left( \frac{G_c A_c (h_f + h_e)}{2h_e^2} - \frac{G_c A_e (h_f + h_e)}{2h_e^2} \right) u_1 - \left( \frac{G_c A_c (h_f + h_e)}{2h_e^2} \right) u_2 \\
+ \left( \frac{G_c A_c (h_f + h_e)}{4h_e^2} - \frac{G_c l_c}{h_e^2} \right) \frac{\partial w_1}{\partial x} + \left( \frac{G_c A_c (h_f + h_e)}{4h_e^2} + \frac{G_c l_c}{h_e^2} + \frac{G_e A_e (h_f + h_e)^2}{4h_e^2} \right) \frac{\partial w_2}{\partial x} = 0
\] (30)

In the present work a BLGNR with clamp-clamp ends is investigated where equations of the boundary condition are given as follow:

\[
u_1 = u_2 = w_1 = w_2 = \frac{\partial w_1}{\partial x} = \frac{\partial w_2}{\partial x} = 0
\] (33)

2.1. Solution Procedure

Because of coupling of the governing equations of motion (equations (23)-(26)), the problem does not have an analytical or semi-analytical solution. For this reason, HDQM [18] is employed. This method was initiated from the idea of conventional integral quadrature and is a numerical discretization technique for the approximation of derivatives [23, 24]. Following this idea, the nth order derivative of the function \( f(x) \) with \( N \) grid points, is approximated by a linear sum of all the functional values in the entire domain, that is,

\[
\frac{\partial^n}{\partial x^n} f(x_i) = \sum_{j=1}^{N} A_{ij}^{(n)} f(x_j), \quad i = 1, 2, ..., N
\] (34)

where \( f(x_i) \) represents the functional value at a grid point \( x_i \), and \( A_{ij}^{(n)} \) is the weighting coefficient of the nth order derivative. For the mesh generation in \( x \) coordinate on the computational domain of BLGNR \( (0 \leq x \leq L) \), Chebyshev distribution method is involved which is described as follow:

\[
x_i = \frac{L}{2} \left[ 1 - \cos \left( \frac{i - 1}{N - 1} \pi \right) \right], \quad i = 1, 2, ..., N
\] (35)

For free vibration analysis of BLGNRs, the dynamic displacement vectors are expressed as follow:

\[
\mathbf{u}(x,t), \mathbf{w}(x,t) = \{U_i(x), W_i(x)\} e^{i\omega t}, \quad i = 1, 2, ..., 6
\] (36)

where \( \omega \) is natural frequency of BLGNR. By substituting equation (37) into equations (23)-(26) and (27)-(32), using \( X = \frac{x}{L} \), \( U_i = \frac{U_i}{h} \) and \( W_i = \frac{W_i}{h} \) as dimensionless parameters, where \( h \) is a carbon atom thickness, and implementing HDQM, the governing equations and boundary conditions are discretized. In order to avoid repetitive representations, only the discretized form of equation (25) is given here, as follows:

\[
\frac{G_c A_c h^2}{E_f A_f h_e^2} \frac{I_c}{L^2 A_c \sum_{k=1}^{N} A_{ik}^{(2)} \tilde{W}_2(X_k)} - \frac{\sum_{k=1}^{N} A_{ik}^{(2)} \tilde{W}_1(X_k)}{2} - \frac{\sum_{k=1}^{N} A_{ik}^{(2)} \tilde{W}_4(X_k)}{2} + \frac{\sum_{k=1}^{N} A_{ik}^{(2)} \tilde{W}_1(X_k)}{2} - \frac{\sum_{k=1}^{N} A_{ik}^{(2)} \tilde{W}_3(X_k)}{2} - \frac{\sum_{k=1}^{N} A_{ik}^{(2)} \tilde{W}_5(X_k)}{2} + \frac{\sum_{k=1}^{N} A_{ik}^{(2)} \tilde{W}_6(X_k)}{2}
\] (37)
\[
\frac{E_c A_e h_c^2}{E_f A_f h_f^2} (W_1(X_i) - W_2(X_i)) + \frac{E_c A_e h_c^2}{E_f A_f h_f^2} W_1(X_i) = \frac{\rho h^2 \omega^2}{E_f} W_1(X_i)
\]

Writing boundary condition and governing equations in matrix form yields the following equations

\[
[A_{BB}][W_B] + [A_{BI}][W_I] = 0 \quad (38)
\]
\[
[A_{IB}][W_B] + [A_{II}][W_I] = \omega^2 [W_I] \quad (39)
\]

where \([W_B]\) and \([W_I]\) are the functional values of the boundary and interior points, respectively. After doing some mathematical simplifications on equations (38) and (39), the following final eigenvalue equation system can be obtained

\[
[A_{II}] - [A_{IB}]^{-1} [A_{BI}][W_I] = \omega^2 [W_I] \quad (40)
\]

Now, the natural frequencies and corresponding mode shapes of BLGNRs can be obtained by solving equation (40).

3. Results and Discussion

In order to study tensile-compressive and shear effects of elastic medium on the natural frequencies of BLGNRs the bending rigidity, mass density, length, thickness of sandwich core, width and thickness of nanoribbon layers are considered to be, respectively: \(D_b = 2.4 \text{ eV}, \ \rho = 2260 \text{ kg m}^{-3}, \ \ L = 10 \text{ nm}, \ \ h_c = 0.335 \text{ nm}, \ \ b = 2 \text{ nm} \ \ \text{and} \ \ h_f = 0.335 \text{ nm}. \) In addition, the equivalent tensile-compressive and shear (in armchair direction) moduli of vdWs interactions between two GNR layers are taken 26.6 GPa and 482 MPa, respectively [1], and the following parameters are defined

\[
E^* = \frac{E_e}{E_c} \quad \text{and} \quad G^* = \frac{G_e}{G_c} \quad (410)
\]

The definitions of \(E^*\) and \(G^*\) say that vdWs interactions between polymer matrix and GNR will be stiffer than those between GNR layers if \(E^* > 1\); and it is the other way round if \(E^* < 1\). It is important to note that \(E_c\) and \(G_c\) are constant in the above definitions.

3.1. Comparison studies

In this section, two comparison studies for natural frequencies are conducted in Tables 1 and 2 to validate the results of the present formulation and confirm its reliability. As the first comparison study, in Table 1 results of the present formulation are compared with the first and second natural frequencies of a CBLGNR with only considering the shear modulus effect of vdWs interactions between GNR layers (the tensile-compressive modulus is considered to be high enough, \(E_c = 4 \text{ TPa}\)) [18]. In the second comparison study (see Table 2) the present results are compared with the out-of-plane/in-phase (OPl-APh) and out-of-plane/anti-phase (OPl-APh) natural frequencies of a simply supported double nano-beam [15] with considering only the tensile-compressive modulus effect of vdWs interactions of GNRs (the shear modulus of vdWs interactions between GNRs is considered to be zero, \(G_c = 0\)) for three different values of the tensile-compressive modulus. As seen from Tables 1 and 2, the results of the present study are in excellent agreement with those reported in literatures.

### Table 1. Comparison of first and second natural frequencies of CBLGNR incorporating the shear modulus effect of vdWs interactions of GNR layers \((E_c = 4 \text{ TPa})\) and \(D_b = 1.4 \text{ eV}\).

<table>
<thead>
<tr>
<th>Shear modulus (GPa)</th>
<th>1st frequency (GHz)</th>
<th>2nd frequency (GHz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ref. [18]</td>
<td>Present study</td>
<td>Ref. [18]</td>
</tr>
<tr>
<td>0.25</td>
<td>6.324</td>
<td>6.324</td>
</tr>
<tr>
<td>4.6</td>
<td>10.246</td>
<td>10.246</td>
</tr>
</tbody>
</table>

### Table 2. Comparison of OPl-APh and OPl-APh dimensionless natural frequencies of BLGNR \((G_c = 0)\).

<table>
<thead>
<tr>
<th>Tensile-compressive modulus (GPa)</th>
<th>First OPl-APh frequency</th>
<th>First OPl-APh frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>9.869</td>
<td>9.869</td>
</tr>
<tr>
<td>20</td>
<td>9.869</td>
<td>9.869</td>
</tr>
<tr>
<td>30</td>
<td>9.869</td>
<td>9.869</td>
</tr>
</tbody>
</table>

3.2. Benchmark Results

In order to investigate the tensile-compressive and shear effects of a surrounding elastic medium on the vibrational behavior of BLGNRs, numerical natural frequency results for different \(E^*\) and \(G^*\) values are given. To this end, frequency ratio is defined as follows:

\[
\text{Frequency ratio} = \frac{\text{Frequency of BLGNR with elastic medium}}{\text{Frequency of BLGNR without elastic medium}}
\]

And five natural frequencies of BLGNR without elastic medium are listed in Table 3. Natural frequencies of BLGNR with elastic medium can be calculated by using Table 3 and frequency ratio values.

In the following, results are presented in three sections. In the first section only tensile-compressive effects and in the second section only shear effects of vdWs interactions of elastic medium
on the first five natural frequencies of BLGNR are investigated.

Table 3. Natural frequencies of BLGNR without elastic medium.

<table>
<thead>
<tr>
<th>Mode number</th>
<th>$\omega_1$ (GHz)</th>
<th>$\omega_2$ (GHz)</th>
<th>$\omega_3$ (GHz)</th>
<th>$\omega_4$ (GHz)</th>
<th>$\omega_5$ (GHz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency</td>
<td>40.61</td>
<td>92.94</td>
<td>163.42</td>
<td>252.01</td>
<td>347.90</td>
</tr>
</tbody>
</table>

In the third section, the tensile-compressive and shear effects of vdWs interactions of elastic medium on natural frequencies are simultaneously investigated. In addition, in the last section effects of aspect ratios of BLGNRs on the natural frequencies are considered for two different values of elastic medium moduli.

![Fig. 3. Variations of frequency ratio with mode number for different values of $E^*$](image_url)

Firstly, in Fig. 3 variations of frequency ratio versus the mode number are plotted for various values of $E^*$ (0.001 < $E^*$ < 10 or 0.0266 < $E_c$ < 266 GPa). It should be noted that in Fig. 3 the value of $G^*$ is set to zero. It is seen from Fig. 3 that all frequency ratio curves show a monotonically decreasing trend as the mode number increases. This implies that the tensile-compressive effects of elastic medium on natural frequencies decrease by increasing the mode number. Furthermore, Fig. 3 shows that the tensile-compressive effects of elastic medium are notably less influential at higher mode numbers. This is due to this fact that as the mode number increases, the homogeneity of BLGNR layers changes from out of plane to in-plane. In general, it can be also seen from Fig. 3 that lower mode numbers are more dependent on the variations of the tensile-compressive modulus value than higher ones. In addition, Fig. 3 displays that the first five natural frequencies of BLGNRs are independent of the value of tensile-compressive modulus for $E^* \geq 0.5$. The reason of this is that increasing the value of $E^*$ causes the elastic medium to become stiffer, and accordingly displacements of BLGNR layers become completely in-plane. As a final point it is worth noting that since by increasing the tensile-compressive modulus of elastic medium displacements of BLGNR layers become completely in-plane, it is expected that natural frequencies of BLGNRs also become independent of the value of interlayer tensile-compressive modulus.

Next, in order to consider the shear modulus effects of the elastic medium on the natural frequencies of BLGNRs Fig. 4 is plotted. In Fig. 4 variations of frequency ratios versus mode number is shown for various values of $G^*$ (0.01 \leq G^* \leq 100 or 0.00482 \leq G_c \leq 48.5 GPa) when $E^* = 0.001$. Fig. 4 shows that all frequency ratio curves have a monotonically decreasing trend as the mode number increases, like the one observed in Fig. 3. Therefore, an important result is that the shear effects of the elastic medium on the natural frequencies decrease by increasing the mode number. Unlike Fig. 3 where variations of the frequency ratios versus the mode number are different for various values of the $E^*$, it can be observed from Fig. 4 that variations of the frequency ratios versus the mode number are the same for various values of the $G^*$. Furthermore, it can be observed from Fig. 4 that by increasing the mode number, variations of the frequency ratios become independent of values of the shear modulus of the elastic medium if $G^* \leq 1$. In another words, effects of the interlayer shear modulus on frequency ratios become more pronounced for $G^* > 1$. A final point to note is that since by increasing the shear modulus of the elastic medium displacements of BLGNR layers become completely out of plane, it is expected that the natural frequencies of BLGNRs also become independent of the value of the interlayer shear modulus.

After considering tensile-compressive and shear moduli effects of the elastic medium on the frequency ratios of BLGNRs separately, now investigating their effects are simultaneously desired. To this end, variations of the frequency ratios versus the mode number for various values of $E^*$ and $G^*$ are shown in Fig. 5. It is seen from Fig. 5 that the tensile-compressive and shear moduli effects of the elastic medium have significant influence on low mode numbers and their effects decrease by increasing the mode numbers. Also, as the stiffness of the elastic medium increases (increasing $E^*$ and $G^*$), the frequency ratio increases. This implies that the natural frequencies of the embedded bilayer nanoribbons will increase by increasing $E^*$ and $G^*$. The final point of Fig. 5 is that for small values of the $E^*$ and $G^*$, variations of the frequency ratios are independent of the mode number.
Comparing Fig. 5 with Figs. 3 and 4 represents that when both the tensile-compressive and shear moduli of elastic medium are simultaneously considered, their effects on natural frequencies significantly increase in comparison with the case that their effects are separately investigated. For example, when \( E^* = 10 \) and \( G^* = 0 \), the first frequency ratio becomes 8.57; and when \( E^* = 0.001 \) and \( G^* = 100 \), the first frequency ratio becomes 3.38. But when \( E^* = 10 \) and \( G^* = 100 \), the first frequency ratio reaches to 54.65. Finally, in Tables 4 and 5 influences of the tensile-compressive and shear moduli of elastic medium are numerically compared with those of interlayer vdWs interactions. In Tables 4 and 5 natural frequencies and their value changes are listed due to considering the elastic medium and interlayer vdWs moduli, respectively. The following findings can be highlighted from comparing the results given in Tables 4 and 5:

- The tensile-compressive modulus of elastic medium has a significant influence on the low mode numbers and its effects become less by increasing the mode numbers whereas the interlayer tensile-compressive modulus of vdWs interactions does not have any effects on low mode numbers, and at higher mode numbers its influence increases as the mode number increases.
- At low mode numbers the influences of the tensile-compressive modulus of the elastic medium are more than those of the shear modulus of the elastic medium while there is no sensible difference between their influences at higher mode numbers. But the interlayer tensile-compressive modulus of vdWs interactions does not have any effects on low mode numbers and its influence is more than the interlayer shear modulus at high mode numbers.
- The major influences of both the interlayer shear modulus and the shear modulus of the elastic medium are at low mode numbers and their effects decrease as the mode number increases.
- When both the tensile-compressive and shear modulus effects are simultaneously considered their influences become more in comparison with the cases that they are separately considered. This is true for both the elastic medium and interlayer vdWs interactions moduli.
- However, the interlayer moduli of vdWs interactions have the highest effects on natural frequencies at low and very high mode numbers, but it can be seen that they have considerable influence on other natural frequencies between the low and very high mode numbers. Whereas the elastic medium moduli effects are only pronounced at low mode numbers.

4. Conclusion

The tensile-compressive and shear effects of vdWs interactions between adjacent GNRs as well as elastic medium and GNRs on free vibration of BGNRs are investigated. To consider in-plane displacements of GNRs and both tensile-compressive and shear effects of vdWs interactions, sandwich beam theory is utilized. Governing equations of motion are obtained and solved numerically by HDQM. Results show that, lower mode numbers are more dependent on the tensile-compressive effects of elastic medium than higher ones. In addition, the tensile-compressive effects of elastic medium on natural frequencies are more than the shear effects of elastic medium especially at low mode numbers. It is also observed that the effects of interlayer shear are pronounced at low mode numbers while it’s the other way round for the effects of interlayer tensile-compressive. This study represents that for an accurate analysis of multi-layer graphene nanoribbons embedded in an elastic medium the tensile-compressive and shear effects of vdWs interactions between adjacent GNRs as well as elastic medium and GNRs must be considered simultaneously.
Table 4. Frequency value and change in frequency value for three different cases of tensile-compressive and shear moduli of elastic medium \(E_s = 26.6 \text{ GPa}, G_c = 0.482 \text{ GPa}\).

<table>
<thead>
<tr>
<th>Mode number</th>
<th>(E^* = 0.01)</th>
<th>(E^* = 0.05)</th>
<th>Change in frequency value (%)</th>
<th>(E^* = 0.001)</th>
<th>(E^* = 0.001)</th>
<th>Change in frequency value (%)</th>
</tr>
</thead>
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<tr>
<td>Frequency (GHz)</td>
<td>(G^* = 0)</td>
<td>(G^* = 0)</td>
<td>(G^* = 0.1)</td>
<td>(G^* = 0.5)</td>
<td>(G^* = 0.1)</td>
<td>(G^* = 0.5)</td>
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<td>107.15</td>
<td>66.29</td>
<td>68.81</td>
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<td>354.75</td>
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Table 5. Frequency value and change in frequency value for three different cases of tensile-compressive and shear moduli of core \(E^* = 0, G^* = 0\).

<table>
<thead>
<tr>
<th>Mode number</th>
<th>(E_s = 26.6 \text{ GPa}, G_c = 42)</th>
<th>(E_s = 4000 \text{ GPa}, G_c = 482)</th>
<th>Change in frequency value (%)</th>
<th>(E_s = 4000 \text{ GPa}, G_c = 48)</th>
<th>Change in frequency value (%)</th>
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<td>(G^* = 48)</td>
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</table>

References


[7] Shahsavari D, Karami B, Janghorban M, Li L. Dynamic characteristics of viscoelastic nanoplates under moving load embedded within visco-


