Transverse Vibration for Non-uniform Timoshenko Nano-beams

K. Torabi*, M. Rahi, H. Afshari

Department of Solid Mechanics, Faculty of Mechanical Engineering, University of Kashan, Kashan, Iran

1. Introduction

The progress of nanotechnology has enthused scientists in their search of producing all sorts of micro/nanostructures such as atomic force microscope cantilever tips, nanowires, nanoactuators, nanopores and nano-beams. Whenever the Euler–Bernoulli beam theory and the Timoshenko beam theory are applied to the analysis of the small beam-like structures, the researchers are found to be inadequate. Being scale free, these classical beam theories could not capture the small scale effect in the mechanical properties. For example, Wang and Hu [1] showed that the classical beam theories are not able to predict the decrease in phase velocities of wave propagation in carbon nano-tubes when the wave number is so large that the nanostructure has a significant influence on the flexural wave dispersion.

In 1972, Eringen [2], Eringen and Edelen [3] and later Eringen [4,5] initiated the nonlocal continuum mechanics to allow for the small scale effect by specifying the stress state at a given point to be a function of the strain states at all points in the body. Then the nonlocal theory of elasticity has been used to study lattice dispersion of elastic waves, wave propagation in composites, dislocation mechanics, fracture mechanics, surface tension fluids, etc. [6-10], and Wang et al. [11] used the nonlocal elasticity constitutive equations to investigate the vibration of carbon nanotubes. Wang et al. [11] neglected the nonlocal effect in writing the shear stress–strain relation of the Timoshenko beam theory (TBT), and therefore the effect of including nonlocal constitutive behavior amounted to using an equivalent shear correction factor.

Reddy [12] used various available beam theories, including the Timoshenko beam theory, are reformulated using the nonlocal differential constitutive
relations of Eringen. Wang et al. [13] analyzed the vibration of nonlocal Timoshenko beams.

This paper presents a formulation of nonlocal elasticity theory for the transverse vibration analysis of non-uniform Timoshenko nano-beams (NUTNB). The differential equations of motion are employed to derive from the variational procedure (Hamilton’s principle) and the solutions numerically calculated using GDQM for pinned-pinned, clamped-damped, clamped-pinned, clamped-free, clamped-slide and pinned-slide boundary conditions. This study makes the first attempt to study the axially distributed forces such as an integration-form varying compression force due to gravity acceleration on NUTNB. In addition, the small scale effects, Timoshenko parameters and axial loading are investigated for non-uniform (tapered) nano-beam. The main purpose of this article is to investigate the vibrational response of NUTNB with axial loading for arbitrary boundary conditions and for various values of the small scale effects. This study is organized as follows. Firstly, in Section 2, the nonlocal elasticity theory of Eringen is presented using the nonlocal constitutive differential equations. Secondly, in Section 3, the governing differential equations describing the transverse vibrations will be derived for NUTNB. Also, the present paper is concerned with the gravity effect on the vibration behavior of nano-beams. Then, in Section 4, the solution procedure and dimensionless equations will be derived from some of the various boundary conditions. In Section 5 the numerical solution of the problem will be solved approximately using GDQM and the matrix form will be transformed for equations of motion. Also in Section 6, the numerical results obtained from the present analysis are discussed in many cases of nano-beam. Finally, in Section 7 conclusions will be drawn and remarks the results of presenting work.

2. Nonlocal Elasticity Theory of Eringen

The classical elasticity theory presents the constitutive equation in the form of an algebraic relationship between the stress and strain tensors while that of the nonlocal elasticity theory according to Eringen [2-5], is that the stress field at a reference point x in an elastic continuum depends not only on the strain at that point but likewise on the strains at all other points in the body. Eringen’s nonlocal elasticity involves spatial integrals which represents weighted averages of the contributions of strain tensors of all points in the body to the stress tensor at the given point. Eringen attributed this fact to the atomic theory of lattice dynamics and experimental observations on phonon dispersion [12]. The scale effects are accounted for in the theory by considering the internal size as a material parameter [14].

2.1. The Constitutive Relations

The basic equations for linear, homogeneous, isotropic, nonlocal elastic solid with zero body force are given by [15],

\[ \sigma = \int K(|x' - x|, \tau) \tau(x') dx' \]  

(1)

where \( \sigma \) is the nonlocal stress tensor, \( \tau(x') \) is the classical, the macroscopic stress tensor at point \( x \) and the kernel function \( K(|x' - x|, \tau) \) are the nonlocal moduli, or attenuation function incorporating into constitutive equations the nonlocal effects at the reference point \( x \) are produced by local strain at the source \( x' \), \( \tau \) is the material constant which is defined as \( \tau = e_0 a/l \) where \( e_0 \) is a constant appropriate to each material, \( a \) is an internal characteristics length (e.g., lattice parameter, granular distance) and \( l \) is an external characteristics length (e.g., crack length, wavelength).

The constitutive Eq. (1) defines the nonlocal constitutive behavior of a Hookean solid and represents the weighted average of the contributions of the strain field of all points in the body to the stress field at a point. Though the integral constitutive relation in Eq. (1) makes mathematical difficulties to obtain the solution of nonlocal elasticity problems, Eringen [4] represents this integral constitutive equation to equivalent differential constitutive equations under certain conditions. For an elastic material in the one dimensional case, the nonlocal constitutive relations may be simplified as [4,13].

2.2. Stress Resultant

The nonlocal constitutive can be approximated to a one-dimensional form, in terms of the strains in the Timoshenko beam theory (TBT) [16],

\[ \sigma_{xx} - \mu \frac{\partial^2 \sigma_{xx}}{\partial x^2} = E \varepsilon_{xx} \]  

(2)

\[ \sigma_{xz} - \mu \frac{\partial^2 \sigma_{xz}}{\partial x^2} = G \gamma_{xz} \]  

(3)

where \( E \) and \( G \) are the Young’s and shear moduli, respectively, and \( \gamma \) is the shear strain. Hence, the nonlocal parameter \( \mu = (e_0 a)^2 \), a in the theory will be led to small-scale effect on the response of structures of nano-size and when \( \mu \) is zero, the constitutive relations will be derived of the local theories.
3. Nonlocal Equations of NUTNB

3.1. The Governing Equations of Motion

The Timoshenko beam theory (TBT), which is based on the displacement field at some points, can be found in [17,18].

All applied loads and geometry are such that the displacements \((u_x, u_y, u_z)\) along the coordinates \((x,y,z)\) are only functions of the \(x\) and \(z\) coordinates and time \(t\). The displacement \(u_z\) is supposed identically zero. The terms \(u\) and \(w\) are the axial and transverse displacements, respectively, of the point \((x,0)\) on the mid-plane (i.e., \(z = 0\)) of the beam and the \(\phi\) denotes the rotation of the cross-section. The nonzero strains according to TBT are expressed as,

\[
\varepsilon_{xx} = \frac{\partial u_x(x,z,t)}{\partial x} + z \frac{\partial \phi(x,t)}{\partial x}
\]

\[
\gamma_{xz} = 2\varepsilon_{xz} = \frac{\partial u_z(x,z,t)}{\partial z} + \frac{\partial u_x(x,z,t)}{\partial x}
\]

\[
\gamma_{zz} = \varphi(x,t) + \frac{\partial w(x,t)}{\partial x}
\]  

(4)

The nonlocal bending moment \(M\), and the shear force \(Q\), can be written in the following form,

\[
M(x,t) = \int_A \sigma_{xx} z dA
\]

(5)

\[
Q(x,t) = \int_A \sigma_{xz} dA
\]

(6)

where \(\sigma_{xx}\) is the normal stress and \(\sigma_{xz}\) the transverse shear stress. \(A\) is the cross-sectional area of the beam. The following relations are introduced for using in the coming sections,

\[
A = \int_A dA , \quad \int_A z dA = 0 , \quad I = \int_A z^2 dA
\]  

(7)

Therefore, the \(x\)-axis is taken along the geometric centric of the beam, where \(I\) is the second moment of area of the cross-section.

In this stage, by multiplying both sides of Eq. (2) by \(z\) and integrating over the cross-section area of the beam, then using Eqs. (4) to (7), the nonlocal Timoshenko constitutive relations yields,

\[
M(x,t) - \mu \frac{\partial^2 M(x,t)}{\partial x^2} = EI(x) \frac{\partial \phi(x,t)}{\partial x}
\]

(8)

Also, by integrating Eq. (3) over the area, and using Eqs. (4) to (7), one obtains,

\[
Q(x,t) - \mu \frac{\partial^2 Q(x,t)}{\partial x^2} = GK_A(x) \left[ \frac{\partial w(x,t)}{\partial x} + \phi(x,t) \right]
\]

(9)

where \(K_s\) denotes the shear correction factor of TBT in order to compensate for the error in assuming a constant shear strain (stress) through the thickness of the beam. The shear correction factors depend not only on the material and geometric parameters but also on the load and boundary conditions.

The shear correction factor is \(9/10\) for a circular shape cross-section and \(5/6\) for a rectangular cross-section [11]. A value of 0.877 was used by Reddy and Pang [19] for the analysis of carbon nanotubes (CNTs) with the relation \(K_s = (5+5\nu)/(6+5\nu)\) for the rectangle and \(K_s = (6+12\nu+6\nu^2)/(7+12\nu+4\nu^2)\) for the circle in which \(\nu\) is Poisson’s ratio with value of \(\nu = 0.3\) [20]. The strain energy \(U\) and the kinetic energy \(T\) of the beam are obtained, respectively, by the following equations [21],

\[
T(t) = \frac{1}{2} \int_0^L \rho A(x) \left[ \frac{\partial u_x(x,t)}{\partial t} \right]^2 + \left[ \frac{\partial u_z(x,t)}{\partial t} \right]^2 \, dx,
\]

(10)

\[
U(t) = \frac{1}{2} \int_0^L \left( \sigma_{xx} \varepsilon_{xx} + \sigma_{xz} \gamma_{xz} \right) dA \, dx
\]

where \(\rho\) is the mass density of the nano-beam material.

By substituting Eqs. (4) to (7), into the above energy statements, and neglecting axial displacement of the neutral web \(u(x,t)\), the kinetic and strain energies with respect to the displacement field may be expressed as,

\[
T(t) = \frac{1}{2} \int_0^L \rho I(x) \left[ \frac{\partial \phi(x,t)}{\partial t} \right]^2 + \rho A(x) \left[ \frac{\partial w(x,t)}{\partial t} \right]^2 \, dx
\]

(11)

\[
U(t) = \frac{1}{2} \int_0^L \left( M(x,t) \frac{\partial \phi(x,t)}{\partial x} + Q(x,t) \frac{\partial w(x,t)}{\partial x} + \varphi(x,t) \right) \, dx
\]

In addition, the work done by the external axial force is denoted by,

\[
W(t) = -\frac{1}{2} \int_0^L P(x) \left[ \frac{\partial w(x,t)}{\partial x} \right]^2 \, dx
\]

(12)

where \(P\) is the distributed axial load along \(x\) axis. The Hamilton’s principle is the most powerful variational principle of mechanics, hence the principle of virtual displacements for the TBT is given by,

\[
\delta \int_0^L \left[ T(t) - U(t) + W(t) \right] \, dt = 0
\]

(13)

Therefore, by substituting Eqs. (11) and (12) into Eq. (13), then, integrating by parts and since \(\delta w\) and \(\delta \phi\) are arbitrary in the domain of nano-beam and then setting the coefficients of \(\delta w\) and \(\delta \phi\) to zero lead to the Euler–Lagrange equations of motion in \(0 < x < L\) as [22].
\[
\frac{\partial M(x,t)}{\partial x} - Q(x,t) = \rho I(x) \frac{\partial^2 \varphi(x,t)}{\partial t^2}
\]
\[
\frac{\partial Q(x,t)}{\partial x} + \frac{\partial}{\partial x} \left[ P(x) \frac{\partial w(x,t)}{\partial x} \right] = \rho A(x) \frac{\partial^2 w(x,t)}{\partial t^2}
\]

The corresponding boundary conditions involve specifying one element of each of the following three pairs at the end of \(x = 0\) and \(x = L\),

\[\varphi(x,t) = 0 \quad \text{or} \quad M(x,t) = 0\]

\[w(x,t) = 0 \quad \text{or} \quad V(x,t) = Q(x,t) + P(x) \frac{\partial w(x,t)}{\partial x} \]

where \(V\) denotes the equivalent shear force. By substituting Eq. (14) into the implicit Eqs. (8) and (9), the explicit expressions of the nonlocal bending moment \(M\) and shear force \(Q\) can be obtained as,

\[M(x,t) = EI \left( \frac{\partial}{\partial x} \left( \frac{\partial \varphi(x,t)}{\partial x} \right) - \mu \left( \frac{\partial}{\partial x} \left( \frac{\partial w(x,t)}{\partial x} \right) \right) \right) \]

\[Q(x,t) = G K A \left( \varphi(x,t) + \frac{\partial w(x,t)}{\partial x} \right) - \mu \left( \frac{\partial}{\partial x} \left( \frac{\partial P(x) \varphi(x,t)}{\partial x} \right) - \rho A(x) \frac{\partial^2 w(x,t)}{\partial t^2} \right) \]

Then, the Euler–Lagrange equations of motion for the nonlocal NUTNB can be derived by inserting Eq. (16) into Eq. (14),

\[\frac{\partial}{\partial x} \left[ E I \left( \frac{\partial}{\partial x} (\varphi(x,t)) - G K A \left( \varphi(x,t) + \frac{\partial w(x,t)}{\partial x} \right) \right) \right] = \rho I(x) \varphi(x,t) - \mu G K A(x) \left( \varphi(x,t) + \frac{\partial w(x,t)}{\partial x} \right) \]

\[\frac{\partial^2}{\partial t^2} \left[ \rho I(x) \varphi(x,t) - \mu G K A(x) \left( \varphi(x,t) + \frac{\partial w(x,t)}{\partial x} \right) \right] = \]

\[\frac{\partial}{\partial x} \left[ G K A(x) \left( \varphi(x,t) + \frac{\partial w(x,t)}{\partial x} \right) \right] = \mu \left( \frac{\partial}{\partial x} \left( \frac{\partial P(x) \varphi(x,t)}{\partial x} \right) - \rho A(x) \frac{\partial^2 w(x,t)}{\partial t^2} \right) \]

### 3.2. The Gravity Acceleration

One kind of the axial force is the influence of gravity. Now in the present study, the transverse vibration characteristics of NUTNB will be analyzed in two cases, with or without axial force. In case with axial force, there are also two cases, one is a NUTNB subjected to the constant axial force and the other is of a cantilever NUTNB under axially distributed gravity force.

Due to the gravity and the self-weight of the nano-beam an integration-form varying compression or tensile force is acting on the nano-beam. The force of non-uniform cantilever nano-beam (Clamped-Free) due to acceleration gravity given by,

\[P_z(x) = -\rho g \int L^0 A(x) \, dx \]

\[P_h(x) = +\rho g \int L^0 A(x) \, dx \]

where compressive force \(P_z\) is used for a standing beam and tensile force \(P_h\) is used for a hanging one.

### 4. The Solution Procedure and Dimensionless Equations

A general solution is assumed in the form,

\[w(x,t) = W(x) e^{i \omega t}, \quad \varphi(x,t) = \Phi(x) e^{i \omega t}, \]

where \(W\) and \(\Phi\) are the amplitude of the generalized displacements and rotation of beam, respectively, \(\omega\) the vibration frequency of NUTNB and \(i = -1\). For convenience and simplification, the governing equations can be expressed in the non-dimensional form by introducing the following dimensionless parameters \([13,14,23]\),

\[\bar{x} = \frac{x}{L}, \quad \bar{w} = \frac{W(x)}{L}, \quad \bar{\varphi} = \frac{\varphi(x,t)}{A_0}, \quad \bar{M} = \frac{ML}{E I_0}, \quad \bar{P}_h = \frac{P_h}{E I_0}, \]

\[\bar{P}_z = \frac{P_z}{L^0 A_0}, \quad \bar{\omega} = \frac{\omega L}{E I_0}, \quad \zeta = \frac{A_0 L^2}{E I_0}, \quad \bar{\xi} = \frac{\alpha_0 L}{E I_0}, \quad \bar{\epsilon} = \frac{\rho A_0 L}{E I_0}, \quad \bar{\mu} = \frac{G K A L}{E I_0}, \]

where \(\bar{\lambda}\) is the non-dimensional natural frequency, \(\zeta\) is the slenderness ratio, \(\alpha\) is the scaling effect parameter of the nano-beam, \(\Omega\) is the shear deformation parameter, \(\bar{\lambda}\) is the non-dimensional bending moment, \(\bar{\mu}\) and \(\bar{\epsilon}\) are the non-dimensional axial and shear forces, respectively, \(\bar{\epsilon}\) is the gravity parameter, \(\bar{\mu}\) and \(\bar{\epsilon}\) are the non-dimensional coefficient and function of the axial force due to gravity, respectively. It should be noted that in the case with gravity, the following statements are used.

\[\bar{\nu} = \frac{\bar{V}}{G K A_0}, \quad \bar{V} = \frac{\bar{P}_h}{G K A_0}, \quad \bar{P}_g = \frac{\bar{P}_z}{L^0 A_0}, \quad \bar{\lambda} = \frac{\lambda L}{E I_0}, \quad \bar{\omega}_g = \frac{\omega L}{E I_0}, \quad \bar{\epsilon}_g = \frac{\rho A_0 L}{E I_0}, \]

\[\bar{\lambda}_g = \frac{\bar{\lambda}_g L}{E I_0}, \quad \bar{\mu}_g = \frac{\bar{\mu}_g L}{E I_0}, \quad \bar{\epsilon}_g = \frac{\rho A_0 L}{E I_0}, \quad \bar{\mu}_g = \frac{G K A L}{E I_0}, \]

\[\bar{\omega}_g = \frac{\bar{\omega}_g L}{E I_0}, \quad \bar{\epsilon}_g = \frac{\rho A_0 L}{E I_0}, \quad \bar{\mu}_g = \frac{G K A L}{E I_0}, \]
\( \mathbf{p} = \mathbf{p}_e \), \( \mathbf{P}(x) = \mathbf{P}_e(x) \) \hspace{1cm} (22)

By substituting Eqs. (20) and (21), into governing equations and corresponding boundary conditions, the nonlocal governing equations for NUTNB in dimensionless form yields,

\[
\frac{\lambda^2 \Omega}{\xi^2} \left[ \hat{T} \hat{\varphi} - \alpha^2 \frac{d^2}{dx^2} \left( \hat{T} \hat{\varphi} \right) \right] + \Omega \frac{d}{dx} \left( \hat{T} \frac{d \hat{\varphi}}{dx} \right) - \hat{\Lambda} \left( \frac{d \hat{\varphi}}{dx} + \hat{\varphi} \right) = 0
\]

\[
\frac{d}{dx} \left[ \hat{\Lambda} \left( \frac{d \hat{\varphi}}{dx} + \hat{\varphi} \right) \right] + \lambda^2 \Omega \left[ \hat{\Lambda} \hat{\varphi} - \alpha^2 \frac{d^2}{dx^2} \left( \hat{\Lambda} \hat{\varphi} \right) \right] - \hat{\nu} \left[ \alpha^2 \frac{d^2}{dx^2} \left( \hat{\nu} \frac{d \hat{\varphi}}{dx} \right) - \frac{d}{dx} \left( \hat{\nu} \frac{d \hat{\varphi}}{dx} \right) \right] = 0
\]

The natural boundary conditions are as,

\[
\hat{\mathbf{M}} = I \left( \frac{d \hat{\varphi}}{dx} + \hat{\varphi} \right) = 0
\]

\[
\hat{\mathbf{V}} = \hat{\Lambda} \left( \frac{d \hat{\varphi}}{dx} + \hat{\varphi} \right) - \lambda^2 \alpha^2 \Omega \frac{d^2}{dx^2} \hat{\Lambda} \hat{\varphi}
\]

\[
- \hat{\nu} \left[ \alpha^2 \frac{d^2}{dx^2} \left( \hat{\nu} \frac{d \hat{\varphi}}{dx} \right) - \frac{d}{dx} \left( \hat{\nu} \frac{d \hat{\varphi}}{dx} \right) \right] = 0
\]

The geometry boundary conditions are as,

\[
\hat{\nu}(\hat{x}) = 0, \quad \hat{\varphi}(\hat{x}) = 0
\] \hspace{1cm} (25)

Note that the nonlocal governing equations given in Eqs. (23) to (25), reduce to that of the counterpart local Timoshenko model when the nonlocal parameter or small scale effect is set to zero (\( \alpha = 0 \)). Finally, the corresponding dimensionless boundary conditions with nano-beam are listed in Table 1.

5. Generalized Differential Quadrature Method (GDQM)

Gauss quadrature is a numerical integration method. Its basic idea is to approximate a definite integral with a weighted sum of integrand values of a group of nodes in the form of,

\[
\int_a^b f(x)dx \approx \sum_{j=1}^{N} w_j f(x_j)
\]

where \( x_j \) are nodes and \( w_j \) are weighting coefficients.

In the differential quadrature method (DQM) the weighting coefficients are determined by solving a system of linear equations. Extending Gauss quadrature to find the derivatives of various orders of a differentiable function gives a rise to the differential quadrature [24,25].

In other words, the derivatives of a function are approximated by weighted sums of the function values in a group of nodes [26,27].

The two major disadvantages of DQM are: the first limits, the small number of grid points and requires solving sets of linear equations, the second limits the distribution of the grid points which is critical in structural dynamic analysis. In GDQM the weighting coefficients for derivative approximations are given by a simple algebraic expression and a recurrence relationship, together with arbitrary choices of grid points [27-30]. Consider the discretization of mth order derivative of \( w(x) \), the following DQ approximation is assumed as,

\[
\frac{\partial^m w(x,t)}{\partial x^m} = \sum_{j=1}^{N} C_m^{(i)} w(x_j,t)
\]

where \( C_m^{(i)} \) are weighting coefficients for mth derivative and \( w(x) \) are function values at grid points \( x_j \) (\( i = 1,2,\ldots,N \)). Therefore, explicit formulas for these coefficients are found to be [26,29],

\[
C_m^{(i)} = \frac{M^{(i)}(x_i)}{(x_i-x_j)M^{(i)}(x_j)}
\]

\[
C_m^{(m)} = \frac{C_m^{(m-1)} - C_m^{(m-1)}}{x_i-x_j}
\]

\[
C_m^{(m)} = -\sum_{j=1}^{N} C_m^{(m)} , i = 1,2,\ldots,N,m = 1,2,\ldots,N-1
\]

By choosing the Lagrange interpolated polynomial \( M(x) \) as the set of test functions, yields,

\[
M(x) = \prod_{j=1}^{N} \left( x - x_j \right) \quad M^{(i)}(x) = \prod_{j=1}^{N} \left( x - x_j \right)
\]

Higher order coefficient matrices at each grid point can be obtained in GDQM from the first order weighting matrix as follows, [31,32],

\[
\frac{d \hat{\nu}(\hat{x})}{d \hat{x}} = \sum_{j=1}^{N} C_m^{(i)} \hat{w}(\hat{x}_j)
\]

\[
\frac{d^2 \hat{\varphi}(\hat{x})}{d \hat{x}^2} = \sum_{j=1}^{N} C_m^{(2)} \hat{\varphi}(\hat{x}_j), \quad \sum_{j=1}^{N} C_m^{(2)} = \sum_{i=1}^{N} C_m^{(1)} C_m^{(1)}
\]

\[
\frac{d^3 \hat{\nu}(\hat{x})}{d \hat{x}^3} = \sum_{j=1}^{N} C_m^{(3)} \hat{\nu}(\hat{x}_j), \quad \sum_{j=1}^{N} C_m^{(3)} = \sum_{i=1}^{N} C_m^{(2)} C_m^{(2)}
\]

\[
\frac{d^4 \hat{\varphi}(\hat{x})}{d \hat{x}^4} = \sum_{j=1}^{N} C_m^{(4)} \hat{\varphi}(\hat{x}_j), \quad \sum_{j=1}^{N} C_m^{(4)} = \sum_{i=1}^{N} C_m^{(3)} C_m^{(3)}
\]

To choose the distribution of the grid points, among non-uniform spacing of nodes which ensure the convergence, the Chebyshev nodes defined by the following equation are nearly optimal [26,27].
Table 1. The dimensionless boundary conditions of NUTNB

<table>
<thead>
<tr>
<th>Type</th>
<th>Boundary conditions (BCs)</th>
<th>Geometry</th>
<th>Natural</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pinned (P)</td>
<td>π = 0</td>
<td>-</td>
<td>M = 0</td>
</tr>
<tr>
<td>Clamped (C)</td>
<td>π = 0, φ = 0</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Free (F)</td>
<td>-</td>
<td>-</td>
<td>φ = 0</td>
</tr>
<tr>
<td>Slide (S)</td>
<td>φ = 0</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

By the expansion of Eqs. (23) and (24), and then rewriting the resultant equations by generalizing differential quadrature (GDQ) model, the governing equations were obtained for the nonlocal NUTNB.

\[
\begin{align*}
\sum_{j=1}^{N} C_{y}^{(1)} \overline{w}_{j} - \overline{A}(\overline{x}) \overline{\phi} + \\
\Omega \left[ \frac{d^{2}T(\overline{x})}{dx^{2}} \sum_{j=1}^{N} C_{y}^{(4)} \overline{\phi}_{j} + T(\overline{x}) \sum_{j=1}^{N} C_{y}^{(2)} \overline{\phi}_{j} \right] + \\
\lambda^{2} \Omega \alpha^{2} \left[ \frac{d^{2}T(\overline{x})}{dx^{2}} \sum_{j=1}^{N} C_{y}^{(4)} \overline{\phi}_{j} + 2 \frac{dT(\overline{x})}{dx} \sum_{j=1}^{N} C_{y}^{(2)} \overline{\phi}_{j} \right] = 0 \\
\overline{A}(\overline{x}) \sum_{j=1}^{N} C_{y}^{(1)} \overline{w}_{j} + \frac{d\overline{A}(\overline{x})}{dx} \sum_{j=1}^{N} C_{y}^{(2)} \overline{w}_{j} - \\
\left[ \frac{d^{2}P(\overline{x})}{dx^{2}} \sum_{j=1}^{N} C_{y}^{(4)} \overline{w}_{j} + 3 \frac{dP(\overline{x})}{dx} \sum_{j=1}^{N} C_{y}^{(2)} \overline{w}_{j} \right] + \\
\alpha^{2} \left[ \frac{d^{2}P(\overline{x})}{dx^{2}} \sum_{j=1}^{N} C_{y}^{(4)} \overline{w}_{j} + \frac{d^{2}P(\overline{x})}{dx^{2}} \sum_{j=1}^{N} C_{y}^{(2)} \overline{w}_{j} \right] - \\
\frac{d^{2}P(\overline{x})}{dx^{2}} \sum_{j=1}^{N} C_{y}^{(4)} \overline{w}_{j} + \frac{d^{2}P(\overline{x})}{dx^{2}} \sum_{j=1}^{N} C_{y}^{(2)} \overline{w}_{j} + \\
A(\overline{x}) \sum_{j=1}^{N} C_{y}^{(4)} \overline{w}_{j} + \frac{dA(\overline{x})}{dx} \sum_{j=1}^{N} C_{y}^{(2)} \overline{w}_{j} + \\
\lambda^{2} \Omega \left[ \overline{A}(\overline{x}) \overline{w} - \alpha^{2} \right] + 2 \frac{dA(\overline{x})}{dx} \sum_{j=1}^{N} C_{y}^{(2)} \overline{w}_{j} = 0
\end{align*}
\]

(32)

Natural boundary conditions may be rewritten in the following statements in the form of GDQM,

\[
\begin{align*}
\overline{M} = -\alpha^{2} \frac{dP(\overline{x})}{dx} \sum_{j=1}^{N} C_{y}^{(1)} \overline{w}_{j} + \\
\frac{\overline{P}(\overline{x})}{\Omega} \sum_{j=1}^{N} C_{y}^{(2)} \overline{w}_{j} \\
+ \overline{I}(\overline{x}) \sum_{j=1}^{N} C_{y}^{(4)} \overline{\phi}_{j} - \\
\lambda^{2} \alpha^{2} \left[ \overline{A}(\overline{x}) \overline{w} + \frac{1}{\xi^{2}} \frac{dI(\overline{x})}{dx} \overline{\phi} + \\
\overline{I}(\overline{x}) \sum_{j=1}^{N} C_{y}^{(4)} \overline{\phi}_{j} + \overline{V} = \overline{A}(\overline{x}) \sum_{j=1}^{N} C_{y}^{(1)} \overline{w}_{j} \right] \\
\overline{\phi} = \left[ \overline{\phi}_{1}, \overline{\phi}_{2}, ..., \overline{\phi}_{N} \right]^{T}
\end{align*}
\]

(33)

where the displacement vector and the rotation vector are expressed as,

\[
\overline{w} = \left[ \overline{w}_{1}, \overline{w}_{2}, ..., \overline{w}_{N} \right]^{T}, \quad \overline{\phi} = \left[ \overline{\phi}_{1}, \overline{\phi}_{2}, ..., \overline{\phi}_{N} \right]^{T}
\]

(34)

Eqs. (32) to (34) can be easily transformed into an eigenvalue problem to obtain the non-dimensional natural frequency. For NUTNB model,

\[
\begin{bmatrix}
K_{11} & K_{12} \\
K_{21} & K_{22}
\end{bmatrix}
\begin{bmatrix}
\overline{w} \\
\overline{\phi}
\end{bmatrix} - \lambda^{2}
\begin{bmatrix}
M_{11} & M_{12} \\
M_{21} & M_{22}
\end{bmatrix}
\begin{bmatrix}
\overline{w} \\
\overline{\phi}
\end{bmatrix} = 0
\]

(35)

In simple form, yields

\[
[K] \{\tau\} - \lambda^{2} [M] \{\tau\} = 0, \quad \{\tau\} = \{\overline{w}, \overline{\phi}\}^{T}
\]

(36)

And K, M are the stiffness and mass matrices, respectively. Also the nonlocal boundary conditions are expressed as,

\[
\overline{M} = [\Omega] \{\tau\} - \lambda^{2} [R] \{\tau\} \\
\overline{V} = [S] \{\tau\} - \lambda^{2} [T] \{\tau\}
\]

(37)

The present problem has been solved for some of the Boundary Conditions (BCs) as: Pinned-Pinned (PP), Clamped-Clamped (CC), Clamped-Pinned (CP), Clamped-Free (CF), Clamped-Slide (CS) and Pinned-Slide (PS). The BCs have been implemented simply using GDQM technique in the following matrix form. To illustrate the technique, Consider the NUTNB pinned at the two ends,
\[ \bar{w} = 0, \bar{\mathbf{M}} = 0, \text{ at } \bar{x} = 0.1. \]  \hspace{1cm} (38)

Eq. (37) is converted to the following form for imposition the natural BCs:
\[ \bar{\mathbf{M}}_0 = \left[ \bar{Q}_0 \right] \{ \bar{\zeta} \} - \bar{\lambda}^2 \left[ \bar{R}_0 \right] \{ \bar{\zeta} \} = 0, \text{ at } \bar{x} = 0 \]
\[ \bar{\mathbf{M}}_1 = \left[ \bar{Q}_1 \right] \{ \bar{\zeta} \} - \bar{\lambda}^2 \left[ \bar{R}_1 \right] \{ \bar{\zeta} \} = 0, \text{ at } \bar{x} = 1 \]  \hspace{1cm} (39)

Therefore, Eq. (36) is transformed to,
\[ \left[ \bar{K}_1 \right] \{ \bar{\zeta} \} - \bar{\lambda}^2 \left[ \bar{M}_1 \right] \{ \bar{\zeta} \} = \{ 0 \} \]  \hspace{1cm} (40)
where the matrices in Eq. (42) have been extracted by omitting the first and \( m \)th row and column, respectively. Hence, using this technique Eq. (42) can easily be reduced to an eigenvalue problem as,
\[ \left[ \bar{K}_1 \right] \{ \bar{\zeta} \} = \bar{\lambda}^2 \{ \bar{\zeta} \} \]  \hspace{1cm} (44)
\[ \left[ \bar{Z}_1 \right] \{ \bar{\zeta} \} = \bar{\lambda}^2 \{ \bar{\zeta} \} \]  \hspace{1cm} (45)

Eq. (44) can be solved by a standard eigenvalue solver, and then the non-dimensional nonlocal natural frequencies of the NUTNB are obtained.

6. Numerical Results and Discussions

6.1. The Technique Verification

Firstly, the bending vibration of a uniform clamped–free Timoshenko beam with the rotary inertia parameter \( r^2 = 0.01 \) and 0.001; and the shear deformation parameter \( s^2 = 2.8 \) and 0.5 [23] is considered by following statements,
\[ \bar{\zeta} = \begin{bmatrix} \bar{w}_1 & \cdots & \bar{w}_{n-1} & \bar{\varphi}_1 & \cdots & \bar{\varphi}_n \end{bmatrix}^T \]  \hspace{1cm} (43)
\[ \bar{\zeta} = \begin{bmatrix} \bar{w}_1 & \cdots & \bar{w}_{n-1} & \bar{\varphi}_1 & \cdots & \bar{\varphi}_n \end{bmatrix}^T \]  \hspace{1cm} (43)

where the matrices in Eq. (42) can be solved by a standard eigenvalue solver, and then the non-dimensional nonlocal natural frequencies of the NUTNB are obtained.

Table 2. The non-dimensional natural frequency \( \bar{\lambda}^2 \) for uniform CF local beam

<table>
<thead>
<tr>
<th>Case</th>
<th>( r^2 = 0.01 ), ( s^2 = 2.8 )</th>
<th>( r^2 = 0.001 ), ( s^2 = 0.5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3.2471</td>
<td>3.2471</td>
</tr>
<tr>
<td>3</td>
<td>32.415</td>
<td>32.4153</td>
</tr>
<tr>
<td>4</td>
<td>49.649</td>
<td>49.6488</td>
</tr>
<tr>
<td>5</td>
<td>65.263</td>
<td>65.2632</td>
</tr>
<tr>
<td>6</td>
<td>70.555</td>
<td>70.5548</td>
</tr>
<tr>
<td>7</td>
<td>84.075</td>
<td>84.0753</td>
</tr>
<tr>
<td>8</td>
<td>92.021</td>
<td>92.0212</td>
</tr>
<tr>
<td>9</td>
<td>105.87</td>
<td>105.8678</td>
</tr>
<tr>
<td>10</td>
<td>113.75</td>
<td>113.7541</td>
</tr>
</tbody>
</table>

Table 3. The effect of gravity on the natural frequencies (Hz) of uniform CF local beam

<table>
<thead>
<tr>
<th>Mode number</th>
<th>( r^2 = 6.30 \times 10^{-5} ), ( s^2 = 2.8 ), ( \bar{\varepsilon} = 0.314 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.2284</td>
</tr>
<tr>
<td>2</td>
<td>1.4513</td>
</tr>
<tr>
<td>3</td>
<td>4.0496</td>
</tr>
<tr>
<td>4</td>
<td>7.8792</td>
</tr>
<tr>
<td>5</td>
<td>12.9026</td>
</tr>
<tr>
<td>6</td>
<td>19.0549</td>
</tr>
<tr>
<td>7</td>
<td>26.2646</td>
</tr>
<tr>
<td>8</td>
<td>34.4547</td>
</tr>
<tr>
<td>9</td>
<td>43.5465</td>
</tr>
<tr>
<td>10</td>
<td>53.4614</td>
</tr>
</tbody>
</table>

Thirdly, the first five dimensionless natural frequencies of a Timoshenko beam are displayed for PP, CP, CC and CF boundary conditions are presented for various small scale effects \( \alpha \) and \( L/d = 10 \), in Table 4. Where the other parameters of beam according to [13] are: diameter \( d = 0.678 \text{ nm} \), lengths \( L = 10d \), and the following assumed mechanical parameters: Young’s modulus \( E = 5.5 \text{ TPa} \), Poisson’s ratio \( \nu = 0.19 \), effective tube thickness \( t = 0.066 \text{ nm} \), shear correction factor \( K_t = 0.563 \). In Table 4, the frequencies have been compared by Wang et al. [13]. Note that the results associated with \( \alpha = 0 \) correspond to the local TBT frequencies. The excellent accuracy and agreement are achieved between the results of the present method and the exact method [13] for local frequencies and also there is a rate of convergence for an adequate number of grid points. Although, for nonlocal frequencies, the result of the proposed technique is very close to the exact solution [13], but there is a critical difference between the present and exact procedures. Wang et al [13] used the constitutive relation to the shear stress and strain as the same as in the local beam theory, however, in the present method according to Ref. [4], the constitutive relation for the shear stress and strain
are considered as the nonlocal beam theory. Hence the present method is more complete and more accurate than Ref. [13].

Finally, the first four frequency parameters of the present method and those obtained by Farchal and Shehli [33] and Chen [34] are shown in Table 5. The rotary inertia parameter \( r^2 = \frac{f}{AL^2} = 0.01 \) and shear deformation parameter \( s^2 = \frac{Er^2}{6K} = 0.03 \) are considered, and also three values of compressive axial load parameter \( PL/EI = 0, 1 \) and 10, are considered. As it can be observed from the Table, the present results yield excellent agreement with those of [33, 34]. The difference between the two sets of results is small and well within 0.0280%. As the compressive axial load is increased, natural frequencies of all modes are reduced.

6.2. The Numerical Examples

In this section, the bending vibration characteristics of the NUTNB will be examined. The effect of the varying cross-section area, small scale, shear deformation, rotational moment of inertia, acceleration gravity and the self-weight of the nano-beam on the non-dimensional natural frequencies and normalized mode shapes is demonstrated. The following parameters, material and geometric properties used in computing the numerical values are \( E = 30 \) MPa [12], \( \rho = 2300 \) kg/m³, \( L = 10 \) nm, \( h = \) varied, \( v = 0.3 \), \( K_s = 5/6 \) and \( g = 9.81 \) m/s².

The problem can be solved for each arbitrary \( A(x) \). In this paper, we assume the \( A(x) \) as,

\[
A(x) = 1 - \frac{x^2}{20f^2}
\]

(47)

To study the effect of nonlocal parameter, frequency ratio is defined as,

\[
\text{Frequency ratio} = \frac{\text{Frequency using nonlocal theory}}{\text{Frequency using local theory}}
\]

(48)

6.2.1. The Convergence of N in GDQM

Minimum number of grid points is performed by the convergence test in GDQM [14] using the following equation required to obtain stable and accurate results. The error percent vs. number of grid points for the fundamental frequency (\( \mu = 0.1 \)) for a Pinned-Pinned beam is shown in Fig. 1. It can be noted form Fig. 1, that eleven number grid points are sufficient in resulting converged solution.

\[
\text{Error percent} = \frac{100 \times (\text{Present result - Converged result})}{\text{Converged result}}
\]

(49)

![Figure 1. The convergence of nonlocal results by GDQM](image)

<table>
<thead>
<tr>
<th>BCs.</th>
<th>Mode number</th>
<th>( \alpha )</th>
<th>( \alpha = 0 )</th>
<th>( \alpha = 0.1 )</th>
<th>( \alpha = 0.3 )</th>
<th>( \alpha = 0.5 )</th>
<th>( \alpha = 0.7 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pinned</td>
<td>1</td>
<td>Present</td>
<td>3.0929</td>
<td>3.0729</td>
<td>3.0170</td>
<td>3.0243</td>
<td>2.6385</td>
</tr>
<tr>
<td>Pinned</td>
<td>2</td>
<td>Present</td>
<td>5.9399</td>
<td>5.9399</td>
<td>5.6257</td>
<td>5.5304</td>
<td>4.0663</td>
</tr>
<tr>
<td>Clamped</td>
<td>1</td>
<td>Present</td>
<td>3.7845</td>
<td>3.7845</td>
<td>3.6846</td>
<td>3.6939</td>
<td>3.1711</td>
</tr>
</tbody>
</table>

Table 4. First five natural frequencies of uniform beam with \( L/d = 10 \)
6.2.2. Effect of Nonlocal Parameter

The non-dimensional local and nonlocal frequencies are computed by using the present GDQM for the NUTNB with $L/h=100$ and various values of small scale for six boundary conditions and the results are listed in Table 6. The effect of the nonlocal parameter $\alpha$ is to reduce the natural frequencies, as it can be seen from the results, which are presented in Table 6.

Hence, as the small scale coefficient increases, the frequencies obtained for the nonlocal beam become smaller than the local counterpart, and so the frequency ratio is always smaller than unity. This reduction is especially significant for the higher vibration modes. Based on the results in Table 6, it is found that for constant nonlocal parameter $\alpha$, among all of the boundary conditions, clamped-clamped contain largest frequency and pinned-slide contain smallest frequency. From Table 6, it is observed that fundamental frequency for damped-free approximately remains unchanged with the increase in the nonlocal parameters. Table 6 implies that the frequencies for non-uniform (tapered) nano-beam are smaller than the same frequencies of uniform nano-beam. The above result can be demonstrated in Figs. 2 to 4.

To demonstrate the effect of higher modes, frequency ratio for various nonlocal parameters is plotted in Figs. 3 and 4, for non-uniform and uniform cross-section, respectively. From these figures, it can be obviously observed that as the nonlocal parameter value increases, the frequency ratio decreases. In addition, at higher mode numbers, all results converge to the local frequency at the higher lengths. This phenomenon is because of the increase in the interactions between atoms at small wavelengths at higher wave numbers [1].

Table 5. The comparison of first four dimensionless natural frequencies for a damped-free uniform beam

<table>
<thead>
<tr>
<th>Axial load</th>
<th>Method</th>
<th>$r^2 = 0.01$</th>
<th>$s^2 = 0.03$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p=0$</td>
<td>Ref. [33]</td>
<td>Mode 1</td>
<td>Mode 2</td>
</tr>
<tr>
<td></td>
<td>1.799</td>
<td>3.820</td>
<td>5.642</td>
</tr>
<tr>
<td></td>
<td>Ref. [34]</td>
<td>1.7864</td>
<td>3.8047</td>
</tr>
<tr>
<td></td>
<td>Present</td>
<td>1.7985</td>
<td>3.8199</td>
</tr>
<tr>
<td>$p=1$</td>
<td>Ref. [33]</td>
<td>1.579</td>
<td>3.715</td>
</tr>
<tr>
<td></td>
<td>1.5768</td>
<td>3.6982</td>
<td>5.5425</td>
</tr>
<tr>
<td></td>
<td>Present</td>
<td>1.5792</td>
<td>3.7145</td>
</tr>
<tr>
<td>$p=10$</td>
<td>Ref. [33]</td>
<td>2.199</td>
<td>4.720</td>
</tr>
<tr>
<td></td>
<td>2.2009</td>
<td>4.7242</td>
<td>6.1811</td>
</tr>
<tr>
<td></td>
<td>Present</td>
<td>2.1993</td>
<td>4.7197</td>
</tr>
</tbody>
</table>

6.2.3. The Effect of Gravity Acceleration

In Table 7, the non-dimensional first five frequency ratios of a standing cantilever nano-beam are listed for uniform & non-uniform cross-section with $A(x) = 1 - 0.05x^2$ and without & under gravity acceleration with gravity parameter $\varepsilon = 0.9025$ and $L/h = 100$.

As Table 7 shows quite clearly, in the case under gravity acceleration, the non-dimensional frequencies are less than the counterpart frequencies in the case without gravity acceleration. Hence the frequencies decrease due to the self-weighting for a standing cantilever beam. Based on the results in this Table, the fundamental frequencies at uniform cross-section are less than the same frequencies at a non-uniform cross-section for both cases under & without gravity (see Fig. 5).

It can be seen from Fig. 6 that the frequency ratio for uniform beam decreases more with increasing nonlocal parameters with respect to the non-uniform beam, and also the frequency ratio for uniform and non-uniform cross-section tends together with the increase of the mode number in standing cantilever nano-beam under gravity.

6.2.4. The Effect of Length

Figs. 7 and 8 show the variation of non-dimensional fundamental frequency ratio with the length of the beam for pinned-pinned and clamped-clamped boundary conditions with non-uniform cross-section $A(x) = 1 - 0.05x^2$.

From Fig. 7(a), it can be comprehended that the space between the curves gradually decreases with the increase in the length of nano-beam. Further, for all curves, frequency ratio converges to one by increasing the length. It is apparent due to the fact that with the increase of length the effect of the non-local parameter decreases and therefore the frequency ratio tends to unity. As expected, this convergence is more rapid for the small values of non-local parameters. These results are similarly found from Fig. 7(b) for clamped-clamped boundary condition.

Fig. 8 shows the small-scale effect on the dimensionless fundamental natural frequency of NUTNB with lengths of $L = 10$, 15 and 20 nm [35]. As it is shown in Fig. 8, the small-scale effects decrease with an increase in the length of a nano-beam.
Table 6. First five non-dimensional frequencies of NUTNB with $L/h = 100$

<table>
<thead>
<tr>
<th>BCs.</th>
<th>Mode number</th>
<th>α</th>
<th>0</th>
<th>0.01</th>
<th>0.02</th>
<th>0.03</th>
<th>0.04</th>
<th>0.05</th>
<th>0.06</th>
<th>0.07</th>
<th>0.08</th>
</tr>
</thead>
</table>

Table 7. First five non-dimensional frequency ratio of standing cantilever nano-beam under gravity acceleration with $L/h = 100$, $\varepsilon = 0.9025$

<table>
<thead>
<tr>
<th>Mode</th>
<th>Type</th>
<th>α</th>
<th>0</th>
<th>0.02</th>
<th>0.04</th>
<th>0.06</th>
<th>0.08</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>G</td>
<td>1.8188</td>
<td>1.8181</td>
<td>1.8118</td>
<td>1.8161</td>
<td>1.8313</td>
<td>1.8128</td>
</tr>
</tbody>
</table>
6.2.5. The Effect of Axial Load

Fig. 9 presents the influence of the tensional axial load on the pinned-pinned non-uniform nano beam with a variation of nonlocal parameters. The fundamental frequency ratio always decreases with the increase in nonlocal parameter; however, this decreasing of restraint will increases the tensional axial load. Further, when the axial load increases the nonlocal and local frequencies tend together, meanwhile, significantly increases the frequencies of NUTNB.

6.2.6. The Normalized Mode Shapes

Unlike for the pinned-pinned beam case where the mode shapes are not affected by the small scale effect parameter, the mode shapes of the clamped-clamped and clamped-pinned beam are significantly influenced by the small scale effect. The similar phenomenon is also found by Wang et al. [13] for linear vibration modes of the nonlocal Timoshenko beams and Yang et al. [36] for nonlinear vibration modes.

Figure 2. The variation of non-dimensional fundamental frequency ratio with the nonlocal parameter for various boundary conditions for (a) non-uniform nano-beam with \( A(x) = 1 - 0.05(x/l)^2 \) and (b) uniform nano-beam

Figure 3. The variation of non-dimensional first five frequency ratio with the nonlocal parameter for NUTNB with \( A(x) = 1 - 0.05(x/l)^2 \) for (a) pinned-pinned, (b) clamped-clamped and (c) clamped-free
7. Conclusions

Transverse vibration characteristics have been investigated for a non-uniform Timoshenko nano-beam based on Eringen’s nonlocal elasticity theory with axially loaded. A numerical approach (GDQM) has been used to study solving governing equations. Many typical results calculated by the presented method show excellent agreement with the exact results by other investigators. The same influence has been studied for the scale effect (nonlocal parameter), varying cross-section area, shear deformation, rotational moment of inertia, size of nano-beam, axial load, acceleration gravity and the self-weight of the NUTNB on the dimensionless natural frequencies. Based on the results, which have been discussed earlier, several conclusions can be addressed as follows:

(1) The small number grid points are sufficient in resulting converged solution and the presented work reflects the power of GDQM in solving non-uniform problems.

(2) The effect of the nonlocal parameter is to reduce the natural frequencies, hence the nonlocal frequencies are smaller than the local counterparts.

(3) The small scale effects are more significant for smaller size of nano-beam, larger nonlocal parameter and higher vibration modes.

(4) Among all of the boundary conditions (PP, CC, CP, CF, CS and PS), clamped-clamped (CC) contain largest frequency and pinned-slide (PS) contains smallest frequency.
(5) The frequency for non-uniform (tapered) nano-beam is smaller than the same frequencies of uniform nano-beam.

(6) Due to gravity acceleration and self-weighting (or distributed compressive axial force) for a standing cantilever beam, the non-dimensional frequencies are less than the counterpart frequencies in the case without gravity acceleration. While the increase in the tensional axial load always causes an increase in the frequencies of NUTNB.

(7) In general, the frequency ratio for uniform beam decreases more by increasing nonlocal parameters with respect to the non-uniform beam for all of the boundary conditions of nano-beam.

(8) The small scale effect parameter is more significant for the mode shapes of the clamped-clamped and clamped-pinned unlike the mode shapes for the pinned-pinned beam case.
**Figure 8.** The variation of non-dimensional fundamental frequency with the nonlocal parameter for various values of length of beam for non-uniform cross-section, (a) pinned-pinned and (b) clamped-clamped

**Figure 9.** The influence of the tensional axial load on pinned-pinned non-uniform nano-beam with the variation of nonlocal parameters

**Nomenclature**

- $\sigma$: nonlocal stress tensor
- $\sigma(x')$: classical, macroscopic stress tensor
- $K$: kernel function
- $\tau$: material constant
- $\varepsilon_0$: constant appropriate to each material
- $a$: internal characteristics length
- $l$: external characteristics length
- $E$: Young's modulus
- $G$: shear modulus
- $\gamma$: shear strain
- $\mu$: nonlocal parameter
- $M$: nonlocal bending moment
- $Q$: shear force
- $\sigma_{xx}$: normal stress
- $\sigma_{xz}$: transverse shear stress
- $A$: cross-sectional area of the beam
- $K_s$: shear correction factor of the TBT
- $\rho$: mass density of the nano-beam
- $P$: distributed axial load along x axis
- $W$: amplitude of the generalized displacements
- $\Phi$: rotation of beam
- $\omega$: vibration frequency of NUTNB
- $\lambda$: natural frequency
- $\zeta$: slenderness ratio
- $\alpha$: scaling effect parameter of the nano-beam
- $\Omega$: shear deformation parameter
- $\bar{M}$: bending moment
- $\bar{p}$: axial forces
- $V$: shear forces
- $\varepsilon$: gravity parameter
- $\bar{p}_g$: coefficient of the axial force due to gravity
- $P_g$: function of the axial force due to gravity
- $x_j$: nodes coefficients.
- $w_j$: weighting coefficients.
- $C_{mij}$: weighting coefficients for mth derivative
- $w(x_j)$: function values at grid points

**References**


