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A Quasi-3D Polynomial Shear and Normal Deformation Theory for Laminated Composite, Sandwich, and Functionally Graded Beams

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ABSTRACT

Bending analyses of isotropic, functionally graded, laminated composite, and sandwich beams are carried out using a quasi-3D polynomial shear and normal deformation theory. The most important feature of the proposed theory is that it considers the effects of transverse shear and transverse normal deformations. It accounts for parabolic variations in the strain/stress produced by transverse shear and satisfies the transverse shear stress-free conditions on the top and bottom surfaces of a beam without the use of a shear correction factor. Variationally consistent governing differential equations and associated boundary conditions are obtained by using the principle of virtual work. Navier closed-form solutions are employed to obtain displacements and stresses for the simply supported beams, which are subjected to sinusoidal and uniformly distributed loads. Results are compared with those derived using other higher-order shear deformation theories. The comparison validates the accuracy and efficiency of the theory put forward in this work.

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1. Introduction

In the last few decades many numerical and classical approaches based on approximate beam theories have been developed by various researchers for the analysis of isotropic and anisotropic beams. The well-known classical beam theory (CBT) developed by Euler and Bernoulli [1] is the simplest theory for the examination of beams, but its application is constrained by its failure to account for the effects of shear and normal deformations. The first-order shear deformation theory (FSDT) of Timoshenko [2] is regarded as an improvement over CBT, but it does not satisfy shear stress conditions on the top and bottom surfaces of a beam and requires a shear correction factor for appropriate explanations of strain energy due to shear deformation. To eliminate the limitations of CBT and FSDT, researchers developed higher-order shear deformation theories (HSDTs). Reddy [3], for example, developed a widely known third-order shear deformation theory for the bending analysis of isotropic and anisotropic beams. Sayyad and Ghugal [4] established a hyperbolic shear deformation theory for the examination of isotropic beams, with consideration for the combined effects of bending rotation and shear rotation. Ghugal and Sharma [5] applied a hyperbolic shear deformation theory, and Ghugal and Waghe [6] used a trigonometric shear deformation theory (TSDT) for the analysis of isotropic beams at various boundary conditions. Sayyad [7] compared various shear deformation theories for investigations into the bending and free vibration of isotropic beams.

Two or more inherently and chemically distinct components—that is, fibers and matrices—form a material called composite material. Composite materials are characterized by improved strength-to-weight and stiffness-to-weight ratios. Nowadays, the use of beams made of composite materials is increasing in fields such as aerospace and aeronautical engineering, navigation, and construction. Accordingly, many researchers have carried out studies on the bending behavior of such beams. Carrera [8] developed a unified formulation for the analysis of laminated composite beams, and Catapano et al. [9] extended this formulation to probe into cross-ply laminated composite beams. Chen et al. [10] constructed a stress model for the FSDT-based anal-

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ysis of laminated composite beams. Gherlone [11] conducted a comparative study of laminated composite and sandwich beams by using the zigzag function in an equivalent single layer theory. Sayyad et al. [12] carried out a flexural analysis of fibrous composite beams by using different refined shear deformation theories based on displacement. Nanda et al. [13] proposed a spectral finite element model by using zigzag theory, and Sayyad et al. [14] presented a simple TSDT for the bending analysis of laminated composite and soft-core sandwich beams. Vo and Thai [15] performed a bending analysis of symmetric and anti-symmetric cross-ply laminated composite beams by adopting a two-variable shear deformation theory, which was further extended by Sayyad et al. [16] for the bending analysis of laminated composite and soft-core sandwich beams. Chakraborti et al. [17] put forward a finite element model grounded in zigzag theory to examine laminated sandwich beams with a soft core. Tonelli et al. [18] carried out a bending analysis of sandwich beams by using an HSDT. Ghugal and Shikhare [19] obtained a general solution for the deflections and stresses of sandwich beams by using a TSDT, and Pawar et al. [20] analyzed the bending of sandwich and laminated composite beams by using a higher-order shear and normal deformation theory.

The use of beams and plates made of functionally graded materials (FGMs) in different engineering fields has recently increased. In a functionally graded beam, material properties gradually change along the spatial direction, thus generating a higher resistance against temperature than that achieved with conventional materials. Giunta et al. [21] analyzed functionally graded beams by using classical and advanced shear deformation theories. Li et al. [22] formulated a general solution for the static and dynamic analysis of functionally graded Timoshenko and Euler beams by extending Levinson's beam theory. Pendhari et al. [23] applied a mixed semi-analytical model for the bending analysis of FGM narrow beams under plane stress conditions. With consideration for warping and shear deformation effects, Benatta et al. [24] inquired into the static analysis of functionally graded beams. Kadoli et al. [25] and Kapuria et al. [26] developed a new HSDT for the bending analysis of FGM beams. A static and dynamic analysis of functionally graded Timoshenko and Euler–Bernoulli beams was carried out by Li [27], with the author considering rotary inertia and shear deformation effects. Ying et al. [28] developed exact solutions for the bending analysis of functionally graded beams resting on an elastic foundation. Sayyad and Ghugal [29] recently developed a unified shear deformation theory for the analysis of functionally graded beams.

1.1 Contributions of the current work

Transverse shear and normal deformations play an important role in the accurate prediction of the structural behavior of beams and plates made of advanced composite materials. Therefore, any refinements to CBTs are generally meaningless unless the effects of transverse shear and normal strains are taken into account. Such effects are neglected in Euler and Bernoulli's CBT [1], FSDT [2], Reddy's parabolic shear deformation theory (PSDT) [3], Touratier's TSDT [30], Soldatos' HSDT [31], Karama et al.'s exponential shear deformation theory (ESDT) [32], and Thai and Vo's theory [33].

Theories that consider the effects of transverse shear and normal deformations are called quasi-3D beam theories. Some of the quasi-3D beam theories discussed in the literature are the non-polynomial shear deformation theories of Sayyad and Ghugal [34], Nguyen et al. [35], Yarasca [36], Mantari and Canales [37], and Osofero et al. [38] and the polynomial shear deformation theory of Vo et al. [39]. A recent initiative by Sayyad and Ghugal [40] involved a review of various beam theories available in the literature for the analysis of isotropic and anisotropic beams.

The use of a non-polynomial shear strain function is computationally more difficult than the adoption of a polynomial shear strain function. The present study therefore extends Murphy's [41] polynomial shear deformation theory by accounting for the effects of thickness stretching (i.e., normal deformation). The quasi-3D theory resulting from this extension is computationally simpler than the other quasi-3D theories cited above. In the theory proposed in the current work, both axial and transverse displacements are functions of x and z coordinates. The theory satisfies the transverse shear strain conditions on the top and bottom surfaces of a beam without the use of a shear correction factor. Governing equations are obtained by using the principle of virtual work and applying a fundamental lemma of calculus. Closed-formed solutions are derived using Navier's solution for simply supported boundary conditions. The accuracy of the theory is confirmed by applying it to bending analyses of advanced composite beams made of isotropic materials, fibrous composite materials, and FGMs. Numerical results are obtained for the simply supported beams, which are subjected to sinusoidal and uniformly distributed loads. The findings are then compared with those in the literature for validation.

2. Problem Formulation

2.1 Beam under consideration: Primary characteristics

Let us consider an advanced composite beam of length L and cross-section area ($b \times h$) in right-hand

Cartesian coordinate systems. The beam occupies region $0 \le x \le L$ in the x-direction, region $-b/2 \le y \le b/2$ in the y-direction, and region $-h/2 \le z \le h/2$ in the z-direction. For simplicity, the width of the beam's cross-section is assumed to be unity. The beam is made of advanced composite materials, and its top surface is subjected to transverse loading.

2.2 Kinematics and constitutive relations

Assuming that u is the displacement of any point in the x-direction and w is the displacement of any point in the z-direction, the following displacement field is derived for the third-order shear and normal deformation theory used in this work:

$$u(x,z)=u_0(x)-z\,w_{0,x}+z\left[1-(4/3)(z/h)^2\right]\phi_x(x)$$
$$w(x,z)=w_0(x)+\left[1-\left(4z^2/h^2\right)\right]\phi_z(x) \qquad (1)$$

where u_0 and w_0 are the displacements of the neutral axis in the x- and z-directions, respectively. ϕ_x and ϕ_z denote the shear slopes. The non-zero strains associated with the theory are obtained from the linear theory of elasticity.

$$\varepsilon_x=u_{0,x}-z\,w_{0,xx}+z\left[1-(4/3)(z/h)^2\right]\phi_{x,x}$$
$$\varepsilon_z=\left(-8z/h^2\right)\phi_z \qquad (2)$$
$$\gamma_{xz}=\left[1-4\left(z^2/h^2\right)\right]\left(\phi_x+\phi_{z,x}\right)$$

where ',x' indicates the derivative with respect to x. The constitutive relations for advanced composite beams are also obtained from the linear theory of elasticity.

$$\begin{Bmatrix}\sigma_X\\ \sigma_z\\ \tau_{xz}\end{Bmatrix}^k=\begin{bmatrix}Q_{11} & Q_{13} & 0\\ Q_{13} & Q_{33} & 0\\ 0 & 0 & Q_{55}\end{bmatrix}^k\begin{Bmatrix}\varepsilon_x\\ \varepsilon_z\\ \gamma_{xz}\end{Bmatrix} \qquad (3)$$

where Q_{ij} are the reduced stiffness coefficients.

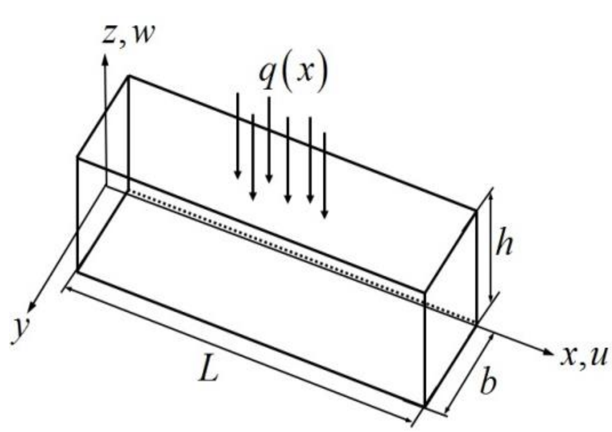

Figure 1. Beam under consideration.

These can be expressed for different materials as follows:

(a) Isotropic material

$$Q_{11}=\frac{E}{1-\mu^2},\quad Q_{13}=\frac{\mu E}{1-\mu^2},\quad Q_{33}=\frac{E}{1-\mu^2},\quad Q_{55}=G \qquad (4)$$

where $\boldsymbol{E}$ denotes the Young's modulus, $\boldsymbol{G}$ represents the shear modulus, and μ is the Poisson's ratio.

(b) Fibrous composite material

$$Q_{11}=\frac{E_1}{1-\mu_{13}\mu_{31}},\qquad Q_{13}=\frac{\mu_{13}E_1}{1-\mu_{13}\mu_{31}},$$
$$Q_{33}=\frac{E_3}{1-\mu_{13}\mu_{31}},\qquad Q_{55}=G_{13} \qquad (5)$$

where E_1 and E_3 are the Young's moduli; μ_{13} and μ_{31} are the Poisson's ratios; and G_{13} is the shear modulus.

(c) FGM

$$Q_{11}=\frac{E(z)}{1-\mu^2},\quad Q_{13}=\frac{\mu E(z)}{1-\mu^2},$$
$$Q_{33}=\frac{E(z)}{1-\mu^2},\quad Q_{55}=\frac{E(z)}{2(1+\mu)} \qquad (6)$$

where,

$$E(z)=E_m+\left(E_c-E_m\right)\left(\frac{1}{2}+\frac{z}{h}\right)^k \qquad (7)$$

where Em and Ec are the Young's moduli of metal and ceramic, respectively, and k is the volume fraction exponent, whose value varies from zero to infinity. The beam is fully ceramic when k is equal to zero and fully metallic when k is infinity.

2.3 Governing differential equations of equilibrium

The governing differential equations of equilibrium can be derived by using the principle of virtual displacements thus:

$$b\int_0^L\int_{-h/2}^{h/2}\left(\sigma_x\delta\varepsilon_x+\sigma_z\delta\varepsilon_z+\tau_{xz}\delta\gamma_{xz}\right)dz\,dx=\int_0^L q\,\delta w\,dx \qquad (8)$$

Substituting the values of stresses and strains from Eqs. (2) and (3) into Eq. (8) and integrating these by parts yield the following governing differential equations:

$$\delta u_0: A_{11}u_{0,xx}-B_{11}w_{o,xxx}+A_{S_{11}}\phi_{x,xx}-A_{C_{13}}\phi_{z,x}=0 \qquad (9)$$

$$\delta w_0: -B_{11}u_{0,xxx}+D_{11}w_{0,xxxx}-B_{S_{11}}\phi_{x,xxx}+B_{C_{13}}\phi_{z,xx}=q \qquad (10)$$

$$\delta\phi_x: A_{S_{11}}u_{0,xx}-B_{S_{11}}w_{0,xxx}+A_{SS_{11}}\phi_{x,xx}+A_{CC_{55}}\phi_x+A_{CC_{55}}\phi_{z,x}-A_{SC_{13}}\phi_{z,x}=0 \qquad (11)$$

$$\delta\phi_z: -A_{C_{13}}u_{0,x}+B_{C_{13}}w_{0,xx}+A_{CC_{55}}\phi_{x,x}-A_{SC_{13}}\phi_{x,x} +A_{CC_{55}}\phi_{x,xx}-A_{DD_{33}}\phi_x=0 \quad (12)$$

where the stiffness coefficients are as follows:

$$\left(A_{ij},B_{ij},D_{ij}\right)=Q_{ij}\int_{-h/2}^{h/2}\left(1,z,z^2\right)dz$$

$$\begin{pmatrix}A_{Sij},B_{Sij},\\ A_{SSij}\end{pmatrix}=Q_{ij}\int_{-h/2}^{h/2}\begin{Bmatrix}f(z),z\,f(z),\\ \left[f(z)\right]^2\end{Bmatrix}dz$$

$$\begin{pmatrix}A_{CCij},A_{Cij},B_{Cij}\\ ,A_{DDij},A_{SCij}\end{pmatrix}=Q_{ij}\int_{-h/2}^{h/2}\begin{Bmatrix}\left[g(z)\right]^2,g'(z),z\,g(z)\\ ,\left[g'(z)\right]^2,g'(z)\,f(z)\end{Bmatrix}dz \quad (13)$$

$$f(z)=z\left[1-(4/3)(z/h)^2\right],$$
$$g(z)=\left[1-4(z/h)^2\right],\ g'(z)=\left(-8z/h^2\right)$$

In this manner, the variationally constant governing differential equations that underlie the theory developed in this study are obtained.

3. Closed-Form Solution

Following Navier's solution procedure, the following solution form is assumed for unknown variables $\left(u_0,w_0,\phi_x,\phi_z\right)$ that satisfy simply supported boundary conditions:

$$\left(u_0\ ,\ \phi_x\right)=\sum_{m=1,3,5,....}^{\infty}\left(u_m\ ,\ \phi_{xm}\right)\cos\left(m\pi x/L\right)$$
$$\left(w_0\ ,\ \phi_z\right)=\sum_{m=1,3,5,....}^{\infty}\left(w_m\ ,\ \phi_{zm}\right)\sin\left(m\pi x/L\right) \quad (14)$$

where $u_m,w_m,\phi_{xm},\phi_{zm}$ are the arbitrary parameters to be determined subject to the condition that the solution in (13) satisfies differential equations (9)–(12). Transverse load q is also expanded in the Fourier sine series as

$$q(x)=\sum_{m=1,3,5....}^{\infty}Q_m\sin\left(m\pi x/L\right)$$
$$Q_m=Q_0 \quad \text{for Single sine load (SSL)} \quad (15)$$
$$Q_m=\frac{4Q_0}{m\pi} \quad \text{for unifrmly distributed load (UDL)}$$

Substituting the solution form from Eqs. (14) and (15) into governing equations (9)–(12) derives

$$\left[K\right]\left\{\Delta\right\}=\left\{f\right\} \quad (16)$$

where [K] is the stiffness matrix, $\left\{\Delta\right\}$ is the vector of unknowns, and $\left\{f\right\}$ is the force vector.

$$\left[K\right]=\begin{bmatrix}A_{11}\alpha^2 & -B_{11}\alpha^3 & A_{S_{11}}\alpha^2 & -A_{C_{13}}\alpha\\ -B_{11}\alpha^3 & D_{11}\alpha^4 & -B_{S_{11}}\alpha^3 & -B_{C_{13}}\alpha^2\\ A_{S_{11}}\alpha^2 & -B_{S_{11}}\alpha^3 & \begin{pmatrix}A_{SS_{11}}\alpha^2\\ +A_{CC_{55}}\end{pmatrix} & \begin{pmatrix}A_{CC_{55}}\\ -A_{SC_{13}}\end{pmatrix}\alpha\\ -A_{C_{13}}\alpha & -B_{C_{13}}\alpha^2 & \begin{pmatrix}A_{CC_{55}}\\ -A_{SC_{13}}\end{pmatrix}\alpha & \begin{pmatrix}A_{CC_{55}}\alpha^2\\ +A_{DD_{33}}\end{pmatrix}\end{bmatrix} \quad (17)$$

$$\left\{\Delta\right\}=\left\{u_m\quad w_m\quad \phi_{xm}\quad \phi_{zm}\right\}^T \text{ and}$$
$$\left\{f\right\}=\left\{0\quad Q_m\quad 0\quad 0\right\}^T \quad (18)$$

4. Illustrative Cases

The developed quasi-3D polynomial shear and normal deformation theory is applied in the bending analyses of advanced composite beams subjected to single sinusoidal and uniformly distributed loads. To confirm the accuracy and validity of the theory, the following cases are solved:

Case 1: Bending analysis of isotropic beams

Case 2: Bending analysis of 0°/90° cross-ply laminated composite beams

Case 3: Bending analysis of 0°/90°/0° cross-ply laminated composite beams

Case 4: Bending analysis of 0°/core/0° sandwich beams

Case 5: Bending analysis of FGMs

The following material properties are used for the detailed numerical study:

MAT 1: $E=210.0$ GPa, $\mu=0.3$, $G=E/2\left(1+\mu\right)$

MAT 2: $E_1=172.4\ GPa, E_3=6.89\ GPa, \mu_{13}=0.25,$ $\mu_{31}=0.01, G_{13}=3.45\ GPa, G_{23}=1.378\,GPa$

MAT 3: $E_1=0.276\ MPa, E_3=3.45\ MPa, \mu_{13}=0.25,$ $\mu_{31}=0.32, G_{13}=0.414\ MPa$

MAT 4: $E_m=70\ GPa, E_c=380\ GPa, \mu_m=0.3,$ $\mu_c=0.3, G=E(z)/2\left(1+\mu\right)$

The numerical results, which are expressed in non-dimensional form, are presented in Tables 1–6 and Figs. 2–13. The various non-dimensional parameters used are as follows:

(a) Isotropic, laminated composite, and sandwich beams

$$\bar{w}\left(\frac{L}{2}\right)=\frac{100E_3h^3}{q_0L^4}w,\quad \bar{u}\left(0,-\frac{h}{2}\right)=\frac{E_3u}{h},$$
$$\bar{\sigma}_x\left(\frac{L}{2},\frac{h}{2}\right)=\frac{\sigma_x}{q_0},\quad \bar{\tau}_{zx}\left(0,0\right)=\frac{\tau_{zx}}{q_0} \quad (19)$$

(b) FGMs

$$\bar{w}\left(\frac{L}{2}\right)=\frac{100E_m h^3 w}{q_0 L^4},\quad \bar{u}\left(0,-\frac{h}{2}\right)=\frac{100E_m h^3 u}{q_0 L^4},$$
$$\bar{\sigma}_x\left(\frac{L}{2},\frac{h}{2}\right)=\frac{h\sigma_x}{q_0 L},\quad \bar{\tau}_{zx}\left(0,0\right)=\frac{h\tau_{zx}}{q_0 L} \tag{20}$$

Case 1: Bending analysis of isotropic beams

In this case, the displacements and stresses of isotropic beams subjected to single sinusoidal and uniformly distributed loads are obtained for aspect ratios (L/h) of 4 and 10. The non-dimensional results are presented in Table 1. The beams are made of an isotropic material MAT 1 (i.e., steel). The findings are compared with the numerical results derived with HSDT [4], PSDT [3], FSDT [2], and CBT [1]. Table 1 shows that the transverse displacement obtained using the proposed theory is of a higher value for an aspect ratio of 4 and produces the exact result for an aspect ratio of 10 compared with the values obtained with PSDT [3]. The stresses obtained for aspect ratios 4 and 10 are in excellent agreement with those derived with other theories for single sinusoidal loads. In the case of isotropic materials, the axial stress is zero at the neutral axis and reaches its maximum at the top and bottom surfaces of the beams. By contrast, the transverse shear stress is at its maximum at the neutral axis and zero at the top and bottom surfaces of the beams. CBT [1] underestimates the deflections and stresses because of this theory's disregard of transverse shear and normal deformations. The same pattern of results is observed for the beam subjected to a uniformly distributed load. Overall, the proposed theory generates excellent results for isotropic beams because of its inclusion of the effects of transverse normal deformations.

Case 2: Bending analysis of 0°/90° cross-ply laminated composite beams

Table 2 presents the results of the comparison of displacements and stresses in two-layer (0°/90°) anti-symmetric laminated composite beams subjected to single sinusoidal and uniformly distributed loads. The layers are of equal thickness, expressed as $h/2$, where $\boldsymbol{h}$ is the overall thickness. The beams are made of fibrous composite materials (MAT 2). The through-thickness variations of axial displacement, axial stress, and transverse shear stress in the two-layer beams are shown in Figs. 2–4. The numerical results are compared with those presented by Reddy [3], Soldatos [31], Karama et al. [32], and Mantari and Canales [37] and those derived using FSDT [2] and CBT [1]. Table 2 indicates that the transverse displacements obtained using the proposed theory are in excellent agreement with those derived with the other quasi-3D polynomial and non-polynomial higher-order theories. FSDT and CBT respectively overestimates and underestimates the transverse displacements because of their neglect of transverse shear and normal deformations. Compared with the values derived with the other higher-order theories, FSDT and CBT generate identical underestimated axial stresses. Transverse shear stresses are obtained using equations of equilibrium to ascertain stress continuity at the layer interface. Figs. 3 and 4 show that the stresses are at their maximum level at the 0° layer—a result attributed to the high elastic modulus along the direction of the fiber in the materials. The stresses are at their minimum at the 90° layer.

Case 3: Bending analysis of 0°/90°/0° cross-ply laminated composite beams

Table 3 illustrates the comparison of the non-dimensional displacements and stresses in three-layer (0°/90°/0°) cross-ply laminated composite beams subjected to single sinusoidal and uniformly distributed loads. The overall thickness (i.e., $h/3$) is equally distributed among all the layers of the beams, which are made of fibrous composite materials (MAT 2). The numerical results are compared with those presented in the literature [1–3, 31, 32, 37]. Table 3 reveals that the transverse deflection of a three-layer laminated beam is less than that of a two-layer (0°/90°) laminated beam. This finding is ascribed to the increase in stiffness along the length of the beams. The displacements and stresses obtained using the quasi-3D theory put forward in this work excellently agree with those derived through the other HSDTs. FSDT and CBT provide overestimated numerical results. The through-thickness variations of axial displacement and stress are shown in Figs. 5 and 6. The figures indicate that because the laminated beams are symmetric, the axial displacement and stress are zero at the neutral axis (i.e., 90° layer) and at their maximum at the top and bottom surfaces of the beam (i.e., 0° layer). The through-thickness variations of transverse shear stress obtained using the equations of equilibrium is shown in Fig. 7.

Case 4: Bending analysis of 0°/core/0° sandwich beams

Sandwich composite beams are constituted by hard face sheets and soft cores. The modulus of the core materials is significantly lower than that of the face sheets. The main benefit of using a sandwich beam lies in its high bending stiffness and high strength-to-weight ratio. Because of these attractive properties, sandwich beam-based structures have been widely used in many industries.

The proposed theory is also validated on the basis of a bending analysis of sandwich beams. The comparison of the numerical results for displacement and stresses in 0°/core/0° sandwich beams subjected to single sinusoidal and uniformly distributed loads is shown in Table 4. Values are obtained for aspect ratios of 4, 10, and 100. The thickness of the face sheets is 0.1 *h*, whereas that of the core is 0.8 *h*. The face sheets are made of MAT 2, whereas the core is composed of MAT 3. The numerical results are compared with those presented by Reddy [3], Soldatos [31], and Karama et al. [32] and those obtained by FSDT [2] and CBT [1]. Table 4 indicates that the central deflection and stresses obtained in the central core are less than those at the top and bottom face sheets. This finding is attributed to the fact that the core is made up of soft transversely isotropic material. The through-thickness variations of axial displacement and stress are shown in Figs. 8–10. As seen in Fig. 9, minimal axial stress is experienced by the core material, thus reflecting that the soft core is resistant only to transverse shear stress.

Case 5: Bending analysis of functionally graded beams

Tables 5 and 6 show the comparison of non-dimensional displacements and stresses in functionally graded beams subjected to single sinusoidal and uniformly distributed loads, respectively. The results on displacements and stresses are obtained for various values of the power-law index (i.e., k = 0, 1, 2, 5, and 10). When k = 0, a beam is fully ceramic. The deflection obtained using the proposed theory is in good agreement with that derived with other higher-order theories. The stresses obtained using the proposed theory are in excellent agreement with the increasing value of k. An increase in the power-law index reduces the stiffness of the functionally graded beams, thereby elevating the displacements and axial stresses. Transverse shear stress decreases with decreasing stiffness of a beam (i.e., increased power-law index). The through-thickness variations of axial displacement and stress are shown in Figs. 11–13. The proposed theory yields a parabolic distribution of transverse shear stress across the depth of the beams and satisfies the zero shear stress conditions on the top and bottom surfaces of the beams (Fig. 12). The axial stress is not zero at the neutral axis, and the transverse shear stress is not at its maximum at such axis. This result is due to the fact that the material properties continuously vary throughout the thickness of the beams.

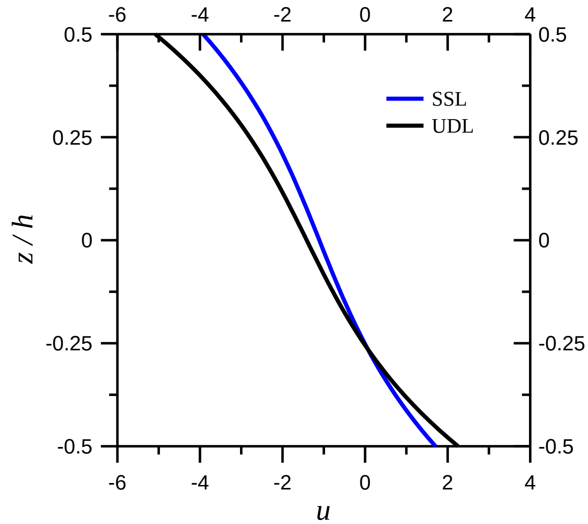

Figure 2. Through-thickness variations of $\bar{u}$ in 0°/90° laminated beams subjected to single sinusoidal and uniformly distributed loading at *L/h = 4*.

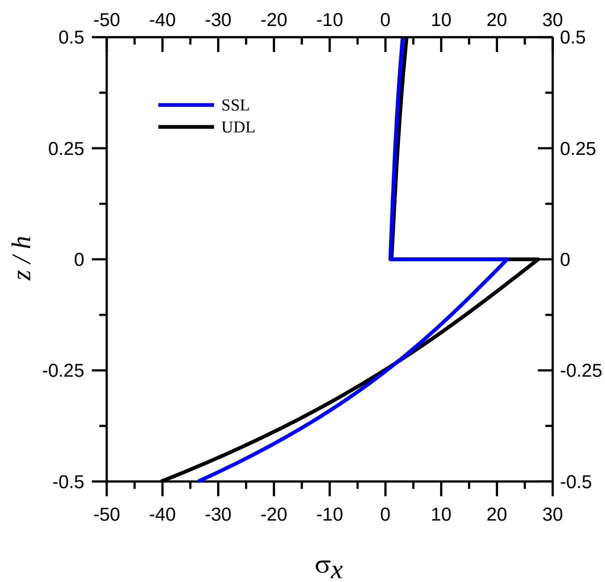

Figure 3. Through-thickness variations of $\bar{\sigma}_x$ in 0°/90° laminated beams subjected to single sinusoidal and uniformly distributed loading at *L/h = 4*.

Table 1. Non-dimensional displacements and stresses in isotropic beams (MAT 1)

h/L	Theory	SSL				UDL			
		$\bar{u}$	$\bar{w}$	$\bar{\sigma}_x$	$\bar{\tau}_{zx}^{EE}$	$\bar{u}$	$\bar{w}$	$\bar{\sigma}_x$	$\bar{\tau}_{zx}^{EE}$
0.25	Proposed	12.248	1.445	9.960	1.897	15.830	1.816	12.135	2.893
	HSDT [4]	12.704	1.427	9.977	1.896	16.486	1.804	12.254	2.882
	PSDT [3]	12.715	1.429	9.986	1.895	16.504	1.806	12.263	2.908
	FSDT [2]	12.385	1.430	9.727	1.910	16.000	1.806	12.000	1.969
	CBT [1]	12.297	1.232	9.727	1.900	16.000	1.563	12.000	-
0.1	Proposed	193.20	1.261	60.98	4.769	249.51	1.599	75.078	7.353
	HSDT [4]	194.31	1.263	61.04	4.769	251.23	1.601	75.259	7.312
	PSDT [3]	194.34	1.264	61.05	4.769	251.27	1.602	75.268	7.361
	FSDT [2]	193.51	1.264	60.79	4.769	250.00	1.602	75.000	4.922
	CBT [1]	192.95	1.232	60.91	4.769	250.00	1.563	75.000	-

Table 2. Non-dimensional displacements and stresses in 0°/90° cross-ply laminated composite beams (MAT 2)

h/L	Theory	SSL				UDL			
		$\bar{u}$	$\bar{w}$	$\bar{\sigma}_x$	$\bar{\tau}_{zx}^{EE}$	$\bar{u}$	$\bar{w}$	$\bar{\sigma}_x$	$\bar{\tau}_{zx}^{EE}$
0.25	Proposed	1.7059	4.4409	33.608	2.9796	2.2524	5.5768	40.2535	5.0407
	PSDT [3]	1.7100	4.4511	33.592	2.9768	2.2580	5.590	40.2390	5.0236
	HSDT [31]	1.6930	4.4039	33.253	2.9513	2.2299	5.533	39.9207	4.8144
	ESDT [32]	1.7450	4.2305	34.264	2.8484	2.3085	5.316	40.9211	5.8468
	Semi-Analytical [23]	---	4.7080	30.019	2.7192	---	5.900	36.6784	3.8488
	HSDT-N1 [37]	1.7066	4.4411	33.5966	2.4774	2.2613	5.5824	40.1544	3.5557
	HSDT-N2 [37]	1.7068	4.4378	33.6027	2.4794	2.2620	5.5789	40.1618	3.5522
	HSDT-N3 [37]	1.7179	4.3931	33.8186	2.5192	2.2745	5.5245	40.3980	3.5999
	FSDT [2]	1.4210	4.7966	27.904	2.9468	1.8360	6.008	34.4272	4.5567
	CBT [1]	1.4210	2.6254	27.904	2.9468	1.8360	3.329	34.4272	4.5567
0.1	Proposed	22.889	2.9158	180.38	7.3604	29.735	3.688	221.260	11.548
	PSDT [3]	22.942	2.9225	180.18	7.3780	29.840	3.696	221.017	11.544
	HSDT [31]	22.901	2.9161	179.86	7.3679	29.7390	3.688	220.692	11.421
	ESDT [32]	23.028	2.8864	180.86	7.3247	29.9363	3.652	221.704	10.698
	Semi-Analytical [23]	----	2.9611	176.53	7.2550	---	3.744	217.330	10.738
	HSDT-N1 [37]	23.1462	2.9495	181.5245	6.2994	30.0738	3.7312	222.6837	9.5100
	HSDT-N2 [37]	23.1429	2.9489	181.6364	6.3082	30.0701	3.7304	222.8253	9.5148
	HSDT-N3 [37]	23.1769	2.9427	181.7649	6.4236	30.1162	3.7229	222.9276	9.6752
	FSDT [2]	22.206	2.9728	174.40	7.3670	28.6882	3.758	215.170	11.391
	CBT [1]	22.206	2.6254	174.40	7.3670	28.6883	3.329	215.170	11.391
0.01	Present	22166	2.6229	17468	73.433	28638.4	3.326	21549.7	113.57
	PSDT [3]	22214	2.6285	17447	73.675	28701.2	3.333	21524.1	113.94
	HSDT [31]	22213	2.6283	17446	73.670	28699.0	3.333	21522.6	113.92
	ESDT [32]	22214	2.6281	17447	73.668	28701.6	3.333	21524.1	113.85
	FSDT [2]	22207	2.6290	17441	73.674	28689.6	3.334	21518.1	113.91
	CBT [1]	22206	2.6254	17440	73.670	28688.2	3.329	21517.0	113.91

Table 3. Non-dimensional displacements and stresses in 0°/90°/0° cross-ply laminated composite beams (MAT 2)

h/L	Theory	SSL				UDL			
		$\bar{u}$	$\bar{w}$	$\bar{\sigma}_x$	$\bar{\tau}_{zx}^{EE}$	$\bar{u}$	$\bar{w}$	$\bar{\sigma}_x$	$\bar{\tau}_{zx}^{EE}$
0.25	Proposed	0.8624	2.700	16.986	1.5561	1.1590	3.367	19.646	1.8346
	PSDT [3]	0.8653	2.700	16.989	1.5570	1.1617	3.368	19.670	1.8310
	HSDT [31]	0.8630	2.698	16.944	1.5594	1.1590	3.365	19.615	1.8312
	ESDT [32]	0.9678	2.687	19.003	1.3320	1.2895	3.366	22.139	1.7557
	Semi-Analytical [23]	---	2.890	18.819	1.5776	---	3.605	21.761	2.4880
	HSDT-N1 [37]	---	---	---	---	---	3.3496	19.6712	---
	HSDT-N2 [37]	---	---	---	---	---	3.3496	19.6784	---
	HSDT-N3 [37]	---	---	---	---	---	3.3852	20.2936	---
	FSDT [2]	0.5136	2.410	10.085	1.7690	0.6636	2.991	12.442	2.7355
	CBT [1]	0.5136	0.510	10.085	1.7690	0.6636	0.648	12.442	2.7355
0.1	Proposed	8.9160	0.873	70.264	4.3342	11.703	1.095	85.098	6.0721
	PSDT [3]	8.9398	0.875	70.212	4.3344	11.733	1.098	85.029	6.0900
	HSDT [31]	8.9329	0.874	70.158	4.3355	11.724	1.097	84.973	6.0922
	ESDT [32]	9.2585	0.889	72.716	4.2051	12.714	1.115	87.629	5.9196
	Semi-Analytical [23]	--	0.933	73.610	4.4390	---	1.170	89.030	6.1500
	HSDT-N1 [37]	---	---	---	---	---	1.0966	85.0144	---
	HSDT-N2 [37]	---	---	---	---	---	1.0970	85.0504	---
	HSDT-N3 [37]	---	---	---	---	---	1.1062	85.6388	---
	FSDT [2]	8.0257	0.814	63.033	4.4226	10.368	1.023	77.767	6.8388
	CBT [1]	8.0257	0.510	63.033	4.4226	10.368	0.648	77.767	6.8388
0.01	Proposed	8018.81	0.513	6319.2	43.999	10361.9	0.651	7794.8	68.243
	PSDT [3]	8034.9	0.514	6310.6	44.217	10382.8	0.652	7784.1	68.243
	HSDT [31]	8034.8	0.514	6310.5	44.217	10382.6	0.652	7784.0	68.244
	ESDT [32]	8038.3	0.514	6313.3	44.204	10388.0	0.653	7786.8	68.046
	FSDT [2]	8025.7	0.514	6303.4	44.226	10368.5	0.651	7776.7	68.387
	CBT [1]	8025.7	0.510	6303.4	44.226	10368.5	0.648	7776.7	68.687

Table 4. Non-dimensional displacements and stresses in 0°/core/0° sandwich beams (Face sheet: MAT 2, Core: MAT 3)

h/L	Theory	SSL				UDL			
		$\bar{u}$	$\bar{w}$	$\bar{\sigma}_x$	$\bar{\tau}_{zx}^{EE}$	$\bar{u}$	$\bar{w}$	$\bar{\sigma}_x$	$\bar{\tau}_{zx}^{EE}$
0.25	Proposed	1.7471	10.052	34.435	1.377	2.3770	12.455	39.429	2.583
	PSDT [3]	1.7393	10.034	34.181	1.372	2.3653	12.494	39.161	2.662
	HSDT [31]	1.7368	10.027	34.132	1.372	2.3616	12.447	39.110	2.655
	ESDT [32]	1.7618	10.045	34.622	1.371	2.3940	12.473	39.647	2.672
	FSDT [2]	1.0120	5.2798	19.898	1.410	1.3080	6.5480	24.549	2.181
	CBT [1]	1.0120	1.0070	19.898	1.410	1.3080	1.2770	24.549	2.181
	Semi-Analytical [23]	---	11.060	37.552	1.356	---	13.750	43.488	2.280
0.1	Proposed	17.706	2.4807	139.55	3.508	23.291	3.0966	168.89	5.305
	PSDT [3]	17.670	2.4772	138.41	3.509	23.24	3.0923	168.13	5.287
	HSDT [31]	17.664	2.4763	138.85	3.509	23.231	3.0911	168.08	5.288
	ESDT [32]	17.731	2.4824	139.38	3.508	23.328	3.0988	168.61	5.286
	FSDT [2]	15.821	1.6910	124.36	3.526	20.439	2.1210	153.43	5.452
	CBT [1]	15.821	1.0070	124.36	3.526	20.439	1.2770	153.43	5.452
	Semi-Analytical [23]	---	2.6680	143.14	3.504	---	3.3300	172.60	5.240
0.01	Proposed	15860	1.0233	12498.6	35.20	20494	1.2973	15416.9	54.42
	PSDT [3]	15839	1.0220	12451.1	35.26	20468	1.2957	15358.4	54.50
	HSDT [31]	15839	1.0219	12451.1	35.26	20468	1.2957	15358.4	54.35
	ESDT [32]	15840	1.0220	12451.7	35.26	20469	1.2958	15358.9	54.49
	FSDT [2]	15820	1.0140	12436.5	35.26	20439	1.2829	15343.3	54.52
	CBT [1]	15821	1.0072	12436.6	35.26	20439	1.2775	15343.5	54.52

Table 5. Non-dimensional displacements and stresses in functionally graded beams under single sinusoidal loading (MAT 4)

k	Theory	L/h = 5				L/h = 20			
		$\bar{u}$	$\bar{w}$	$\bar{\sigma}_x$	$\bar{\tau}_{zx}$	$\bar{u}$	$\bar{w}$	$\bar{\sigma}_x$	$\bar{\tau}_{zx}$
0	Proposed	0.9150	3.1397	3.8341	0.7230	0.2302	2.8947	15.0719	0.7376
	Li et. al [22]	0.9402	3.1657	3.8020	0.7500	0.2306	2.8962	15.0130	0.7500
	TBT [33]	0.9398	3.1654	3.8020	0.7332	0.2306	2.8962	15.0129	0.7451
	SBT [33]	0.9409	3.1649	3.8053	0.7549	0.2306	2.8962	15.0138	0.7686
	HBT [33]	0.9397	3.1654	3.8017	0.7312	0.2306	2.8962	15.0129	0.7429
	EBT [33]	0.9420	3.1635	3.8083	0.7763	0.2306	2.8961	15.0145	0.7920
	Vo et al. [39]	---	3.1397	3.8005	0.7233	---	2.8947	15.0125	0.7432
	HSDT2 [36]	---	3.1397	3.8028	0.7235	---	2.8947	15.0197	0.7443
	HSDT3 [36]	--	3.1397	3.8021	0.7224	--	2.8947	15.0195	0.7433
	CBT [1]	0.9211	2.8783	3.7500	---	0.2303	2.8783	15.0000	---
1	Proposed	2.1975	6.1338	5.7941	0.7230	0.5517	5.7201	23.2714	0.7376
	Li et. al [22]	2.3045	6.2599	5.8837	0.7500	0.5686	5.8049	23.2054	0.7500
	TBT [33]	2.3038	6.2594	5.8836	0.7332	0.5686	5.8049	23.2053	0.7451
	SBT [33]	2.3058	6.2586	5.8892	0.7549	0.5686	5.8049	23.2067	0.7686
	HBT [33]	2.3036	6.2594	5.8831	0.7312	0.5685	5.8049	23.2052	0.7429
	EBT [33]	2.3075	6.2563	5.8943	0.7763	0.5686	5.8047	23.2080	0.7920
	Vo et al. [39]	---	6.1338	5.8812	0.7233	---	5.7201	23.2046	0.7432
	HSDT2 [36]	---	6.1334	5.8855	0.7235	---	5.7197	23.2184	0.7443
	HSDT3 [36]	--	6.1334	5.8843	0.7224	--	5.7197	23.2181	0.7433
	CBT [1]	2.2722	5.7746	5.7959	---	0.5680	5.7746	23.1834	---
2	Proposed	2.9460	7.8606	6.6179	0.6620	0.7397	7.2805	27.2030	0.6757
	Li et. al [22]	3.1134	8.0602	6.8812	0.6787	0.7691	7.4415	27.0989	0.6787
	TBT [33]	3.1130	8.0677	6.8826	0.6706	0.7691	7.4421	27.0991	0.6824
	SBT [33]	3.1153	8.0683	6.8901	0.6933	0.7692	7.4421	27.1010	0.7069
	HBT [33]	3.1127	8.0675	6.8819	0.6685	0.7691	7.4420	27.0989	0.6802
	EBT [33]	3.1174	8.0667	6.8969	0.7157	0.7692	7.4420	27.1027	0.7315
	Vo et al. [39]	---	7.8606	6.8818	0.6622	---	7.2805	27.0988	0.6809
	HSDT2 [36]	---	7.8598	6.8871	0.6625	---	7.2797	27.1158	0.6800
	HSDT3 [36]	--	7.8597	6.8857	0.6613	--	7.2797	27.1154	0.6790
	CBT [1]	3.0740	7.4003	6.7676	---	0.7685	7.4003	27.0704	---
5	Proposed	3.5050	9.6038	7.9579	0.5838	0.8797	8.6479	31.9586	0.5966
	Li et. al [22]	3.7089	9.7802	8.1030	0.5790	0.9133	8.8151	31.8112	0.5790
	TBT [33]	3.7100	9.8281	8.1106	0.5905	0.9134	8.8182	31.8130	0.6023
	SBT [33]	3.7140	9.8367	8.1222	0.6155	0.9134	8.8188	31.8159	0.6292
	HBT [33]	3.7097	9.8271	8.1095	0.5883	0.9134	8.8181	31.8127	0.5998
	EBT [33]	3.7177	9.8414	8.1329	0.6404	0.9135	8.8191	31.8185	0.6562
	Vo et al. [39]	---	9.6037	8.1140	0.5840	---	8.6479	31.8137	0.6010
	HSDT2 [36]	---	9.6030	8.1202	0.5843	---	8.6471	31.8341	0.6019
	HSDT3 [36]	--	9.6025	8.1184	0.5829	--	8.6471	31.8337	0.6014
	CBT [1]	3.6496	8.7508	7.9428	---	0.9124	8.7508	31.7711	---
10	Proposed	3.6922	10.7578	9.6903	0.6394	0.9267	9.5749	37.9164	0.6534
	Li et. al [22]	3.8860	10.8979	9.7063	0.6436	0.9536	9.6879	38.1372	0.6436
	TBT [33]	3.8864	10.9381	9.7122	0.6467	0.9536	9.6905	38.1385	0.6596
	SBT [33]	3.8913	10.9420	9.7238	0.6708	0.9537	9.6908	38.1414	0.6858
	HBT [33]	3.8859	10.9375	9.7111	0.6445	0.9536	9.6905	38.1383	0.6572
	EBT [33]	3.8957	10.9404	9.7341	0.6944	0.9538	9.6907	38.1440	0.7115
	Vo et al. [39]	---	10.7578	9.7164	0.6396	---	9.5749	38.1395	0.6583
	HSDT2 [36]	---	10.7573	9.7234	0.6399	---	9.5742	38.1624	0.6614
	HSDT3 [36]	--	10.7569	9.7215	0.6386	--	9.5743	38.1636	0.6529
	CBT [1]	3.8097	9.6072	9.5228	---	0.9524	9.6072	38.0913	---

Table 6. Non-dimensional displacements and stresses in functionally graded beams under uniformly distributed loading (MAT 4)

k	Theory	L/h = 5			
		$\bar{u}$	$\bar{w}$	$\bar{\sigma}_x$	$\bar{\tau}_{zx}$
0	Proposed	0.7086	2.5047	3.1048	0.4769
	PSDT [3]	0.7251	2.5020	3.0916	0.4769
	TSDT [29]	0.7259	2.5016	3.0949	0.4920
	HSDT [29]	0.7247	2.5003	3.0899	0.4739
	ESDT [29]	0.7280	2.4974	3.1039	0.4871
	FSDT [2]	0.7129	2.5023	3.0396	0.3183
	CBT [1]	0.7129	2.2693	3.0396	----
1	Proposed	1.7051	4.8435	5.0392	0.4769
	PSDT [3]	1.7793	4.9458	4.7856	0.5243
	TSDT [29]	1.7806	4.9451	4.7912	0.5331
	HSDT [29]	1.7517	4.9257	4.7165	0.6025
	ESDT [29]	1.7819	4.9432	4.7944	05430
	FSDT [2]	1.7588	4.8807	4.6979	0.5376
	CBT [1]	1.7588	4.5228	4.6979	----
5	Proposed	2.7143	7.5938	6.9216	0.3856
	PSDT [3]	2.8644	7.7723	6.6057	0.5314
	TSDT [29]	2.8671	7.7792	6.6172	0.5144
	HSDT [29]	2.8641	7.7715	6.6047	0.5332
	ESDT [29]	2.8697	7.7830	6.6281	0.5022
	FSDT [2]	2.8250	7.5056	6.4382	0.9942
	CBT [1]	2.8250	6.8994	6.4382	----
10	Proposed	2.8591	8.5088	8.2877	0.4224
	PSDT [3]	2.9989	8.6530	7.9080	0.4226
	TSDT [29]	3.0022	8.6561	7.9195	0.4392
	HSDT [29]	2.9986	8.6527	7.9070	0.4211
	ESDT [29]	3.0054	8.6547	7.9301	0.4558
	FSDT [2]	2.9488	8.3259	7.7189	1.2320
	CBT [1]	2.9488	7.5746	7.7189	----

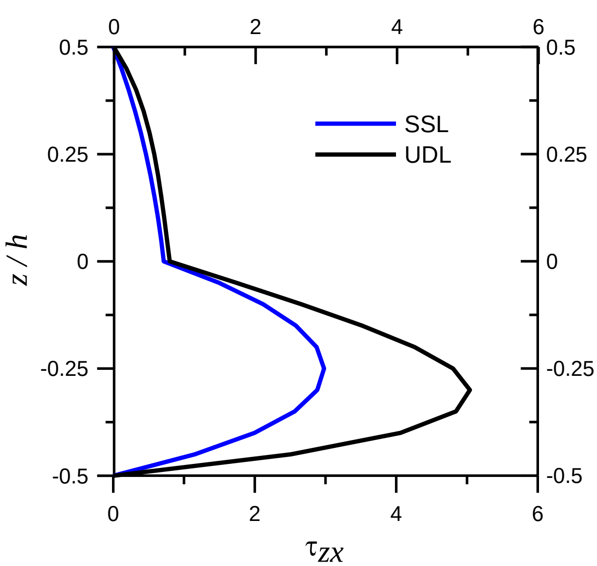

Figure 4. Through-thickness variations of $\bar{\tau}_{zx}$ in 0°/90° laminated beams subjected to single sinusoidal and uniformly distributed loading at *L/h = 4*.

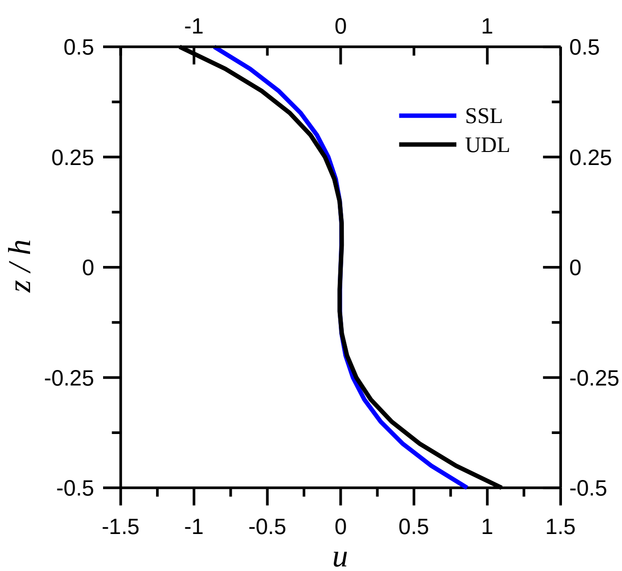

Figure 5. Through-thickness variations of $\bar{u}$ in 0°/90°/0° laminated beams subjected to single sinusoidal and uniformly distributed loading at *L/h = 4*.

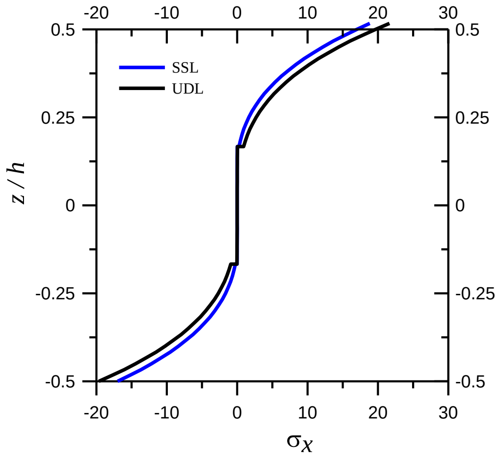

Figure 6. Through-thickness variations of $\bar{\sigma}_x$ in 0°/90°/0° laminated beams subjected to single sinusoidal and uniformly distributed loading at *L/h = 4*.

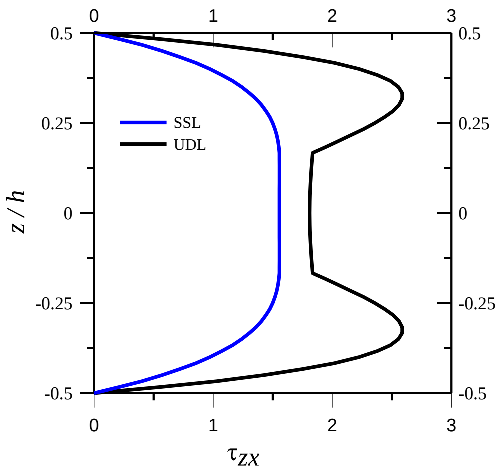

Figure 7. Through-thickness variations of $\bar{\tau}_{zx}$ in 0°/90°/0° laminated beams subjected to single sinusoidal and uniformly distributed loading at *L/h = 4*.

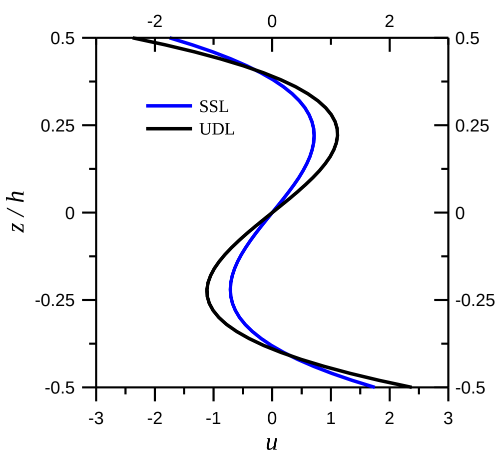

Figure 8. Through-thickness variations of $\bar{u}$ in sandwich beams subjected to single sinusoidal and uniformly distributed loading at *L/h = 4*.

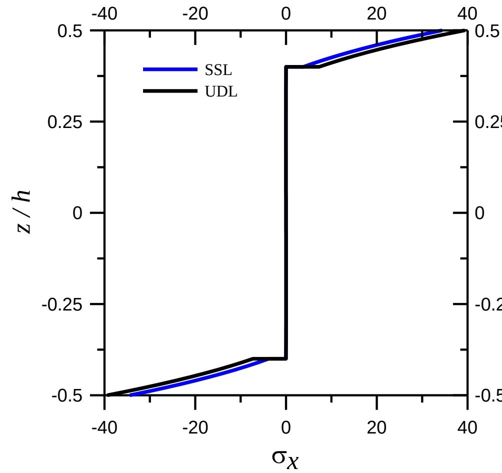

Figure 9. Through-thickness variations of $\bar{\sigma}_x$ in sandwich beams subjected to single sinusoidal and uniformly distributed loading at *L/h = 4*.

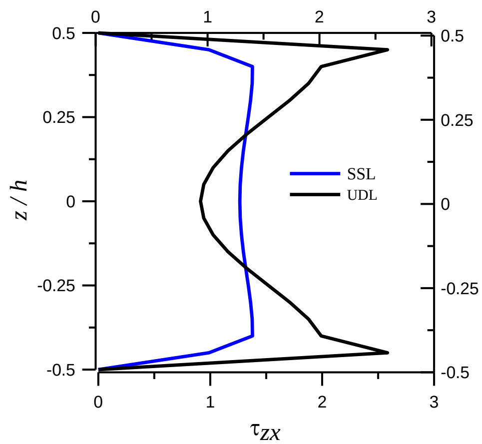

Figure 10. Through-thickness variations of $\bar{\tau}_{zx}$ in sandwich beams subjected to single sinusoidal and uniformly distributed loading at *L/h = 4*.

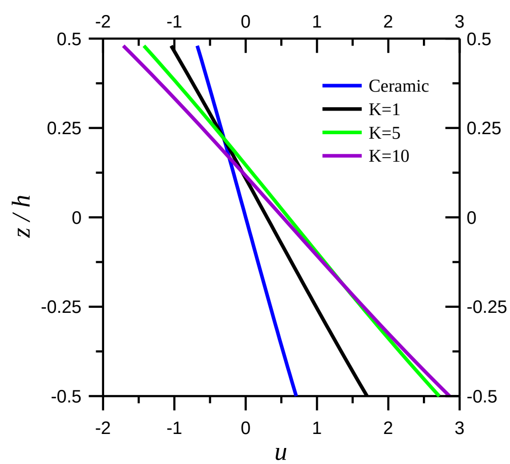

Figure 11. Through-thickness variations of $\bar{u}$ in functionally graded beams subjected to single sinusoidal loading at *L/h = 4*.

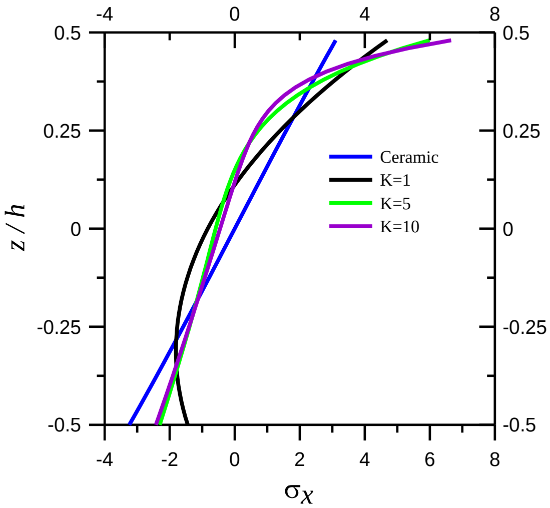

Figure 12. Through-thickness variations of $\bar{\sigma}_x$ in functionally graded beams subjected to single sinusoidal loading at *L/h = 4*.

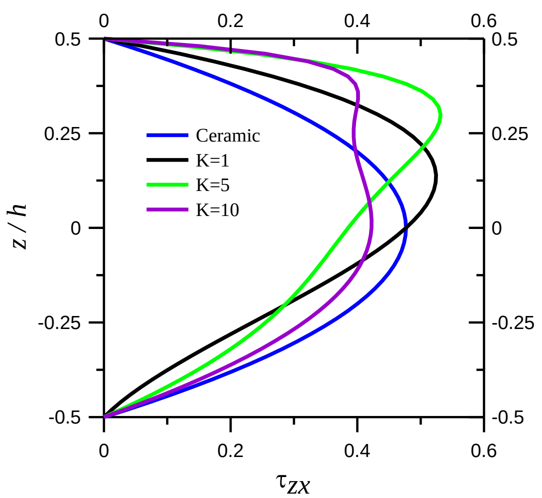

Figure 13. Through-thickness variations of $\bar{\tau}_{zx}$ in functionally graded beams subjected to single sinusoidal loading at *L/h = 4*.

5. Concluding Remarks

In this research, a quasi-3D polynomial shear and normal deformation theory is applied for the bending analyses of composite beams made of fibrous composite materials and FGMs. The proposed theory considers the effects of transverse shear and normal deformations. It also satisfies the traction-free conditions on the top and bottom surfaces of beam without the application of a shear correction factor. Governing equations are obtained using the virtual work principle, and displacements and stresses are determined using Navier's solution. Numerical results are presented for isotropic, laminated composite, sandwich, and functionally graded beams. On the basis of the findings, we can conclude that the proposed theory derives excellent results on displacements and stresses for the examined beams. Shear stress continuity is satisfied by equations of equilibrium. The transverse displacement obtained using the proposed theory for functionally graded beams increases with increasing power-law index given the fact that an increase in the index improves the flexibility of functionally graded beams.

References

[1] Bernaulli J. Curvatura laminae elasticae, Acta, *Eruditorum Liipsiae*. 1964; 262-276.

[2] Timoshenko SP. on the correction for shear of the differential equation for transverse vibrations of prismatic bar. *Philosophical Magazine Series 6.* 1921; 41: 744-746.

[3] Reddy JN. Nonlocal theories for bending, buckling and vibration of beams. *Int J Eng Sci* 2007; 45: 288–307.

[4] Sayyad AS, Ghugal YM. Flexure of thick beams using new hyperbolic shear deformation theory. *Int J Mech* 2011; 5: 113-122.

[5] Ghugal YM, Sharma R. A hyperbolic shear deformation theory for flexure and vibration of thick isotropic beams. *Int J Comput Meth* 2009; 6(4): 585-604.

[6] Ghugal YM, Waghe UP. Flexural analysis of deep beams using trigonometric shear deformation theory. *IEI (India) J* 2011; 92: 3-9.

[7] Sayyad A.S. Static flexure and free vibration analysis of thick isotropic beams using different higher order shear deformation theories. *Int J Appl Math Mech* 2012; 8(14): 71-87.

[8] Carrera E. Giunta G. Refined beam theories based on a unified formulation. *Int J Appl Mech* 2010; 2(1): 117–143.

[9] Catapano A. Giunta G. Belouettar S. Carrera E. Static analysis of laminated beams via a unified formulation. *Compos Struct* 2011;94:75–83.

[10] Chen W. Li L. Xua M. A modified couple stress model for bending analysis of composite laminated beams with first order shear deformation. *Compos Struct* 2011; 93: 2723–2732.

[11] Gherlone M, Tessler A, Sciuva MD. C^0 beam elements based on the refined zigzag theory for multilayered composite and sandwich laminates. *Compos Struct* 2011;93:2882–2894.

[12] Sayyad AS, Ghugal YM, Borkar RR. Flexural analysis of fibrous composite beams under various mechanical loadings using refined shear deformation theories. *Compos: Mech Comput Appl An Int J* 2014; 5(1): 1–19.

[13] Nanda N, Kapuria S, Gopalakrishnan S. Spectral finite element based on an efficient layerwise theory for wave propagation analysis of composite and sandwich beams. *J Sound Vib* 2014; 333: 3120–3137.

[14] Sayyad AS, Ghugal YM, Naik NS. Bending analysis of laminated composite and sandwich beams according to refined trigonometric beam theory. *Curved Layered Struct* 2015; 2: 279–289.

[15] Vo TP, Thai HT. Static behavior of composite beams using various refined shear deformation theories. *Compos Struct* 2012; 94: 2513–2522.

[16] Sayyad AS, Ghugal YM, Shinde PN. Stress analysis of laminated composite and soft core sandwich beams using a simple higher order shear deformation theory. *J Serb Soc Comput Mech* 2015; 9(1): 15-35.

[17] Chakrabarti A, Chalak HD, Iqbal MA, Sheikh AH. A new FE model based on higher order zigzag theory for the analysis of laminated sandwich beam with soft core. *Compos Struct* 2011; 93: 271–279.

[18] Tonelli D, Bardella L, Minelli M. A critical evaluation of mechanical models for sandwich beams. *J Sand Struct Mater* 2012; 14(6): 629–654.

[19] Ghugal YM, Shikhare GU, Bending analysis of sandwich beams according to refined trigonometric beam theory. *J Aerospace Eng Technol* 2015; 5(3): 27-37.

[20] Pawar EG, Banerjee S, Desai YM. Stress analysis of laminated composite and sandwich beams using a novel shear and normal deformation theory. *Lat Am J Solids Struct* 2015; 12:134-161.

[21] Giunta G, Crisafulli D, Belouettar S, Carrera E. A thermo-mechanical analysis of functionally graded beams via hierarchical modeling. *Compos Struct* 2013; 95: 676–690.

[22] Li XF, Wang BL, Han JC. A higher-order theory for static and dynamic analyses of functionally graded beams. *Arch Appl Mech* 2010; 80: 1197–1212.

[23] Pendhari SS, Kant T, Desai YM, Subbaiah CV. On deformation of functionally graded narrow beams under transverse loads. *Int J Mech Mater Des* 2010; 6: 269–282.

[24] Benatta MA, Mechab I, Tounsi A, Bedia EAA. Static analysis of functionally graded short beams including warping and shear deformation effects. *Comput Mater Sci* 2008; 44: 765–773.

[25] Kadoli R, Akhtar K, Kadoli NG. Static analysis of functionally graded beams using higher order shear deformation theory. *Appl Math Model* 2008; 32: 2509–2525.

[26] Kapuria S, Bhattacharyya M, Kumar AN. Bending and free vibration response of layered functionally graded beams: A theoretical model and its experimental validation. *Compos Struct* 2008; 82: 390–402.

[27] Li XF. A unified approach for analyzing static and dynamic behaviors of functionally graded Timoshenko and Euler–Bernoulli beams *J Sound Vib* 2008; 318: 1210–1229.

[28] Ying J, Lu CF, Chen WQ. Two-Dimensional elasticity solutions for functionally graded beams resting on elastic foundations. *Compos Struct* 2008; 84: 209–219.

[29] Sayyad AS, Ghugal YM. A unified shear deformation theory for the bending of isotropic, functionally graded, laminated and sandwich beams and plates. *Int J Appl Mech* 2017; 9(1): 1-36.

[30] Touratier M. An efficient standard plate theory. *Int J Eng Sci* 1991; 29(8): 901–916.

[31] Soladatos KP. A transverse shear deformation theory for homogeneous monoclinic plates. *Acta Mech* 1992;94: 195-200.

[32] Karama M, Afaq KS, Mistou S. Mechanical behavior of laminated composite beam by new multi-layered laminated composite structures model with transverse shear stress continuity. *Int J Solids Struct* 2003; 40: 1525-1546.

[33] Thai HT, Vo TP. Bending and free vibration of functionally graded beams by using various higher order shear deformation beam theories. *Int J Mechanical Science*. 2012; 62: 57-66.

[34] Sayyad AS, Ghugal YM. Effect of transverse shear and transverse normal strain on bending analysis of cross-ply laminated beams. *Int J Appl Math Mech* 2011; 7(12): 85-118.

[35] Nguyen TK, Vo TP, Nguyen BD, Lee J. An analytical solution for buckling and vibration analysis of functionally graded sandwich beams using a quasi-3D shear deformation theory. *Compos Struct* 2016; 156: 238–252.

[36] Yarasca J, Mantari JL, Arciniega RA. Hermite–Lagrangian finite element formulation to study functionally graded sandwich beams. *Compos Struct* 2016; 140: 567–581.

[37] Mantari JL, Canales FG. Finite element formulation of laminated beams with capability to model the thickness expansion. *Compos Part B* 2016; 101:107-115.

[38] Osofero AI, Vo TP, Nguyen TK, Lee J. Analytical solution for vibration and buckling of functionally graded sandwich beams using various quasi-3D theories. *J Sand Struct Mater* 2015; 1–27 (In press).

[39] Vo TP, Thai HT, Nguyen TK, Inam F, Lee J. Static behaviour of functionally graded sandwich beams using a quasi-3D theory. *Compos Part B* 2015; 68: 59–74.

[40] Sayyad AS, Ghugal YM. Bending, buckling and free vibration of laminated composite and sandwich beams: A critical review of literature. *Compos Struct* 2017; 171: 486–504
[41] Murty K, Vellaichamy S. Higher-order theory of homogeneous plate flexure. *AIAA J* 1988; 26: 719-725.